Control Theory and Hyperbolic Partial Differential Equations

PhD lecture
Doctorate school Mathematics and Models

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The aim of this course is to offer a training in *nonlinear hyperbolic equations* of *Hamilton-Jacobi* kind, together with a range of more specialized options, providing students with the basic tools necessary for the comprehension of some actual subjects of research in applied mathematics.

The HJB approach we are concerned with is a technique used to study optimal control problems, and it is based on a functional equation known as the *Dynamic Programming Principle*, introduced by Richard Bellman in the 1950s. This functional equation holds under very mild hypotheses and it is basically an optimality condition that suggests that some quantity remains constant all along optimal trajectories of the dynamic optimization problem at hand.

The main advantage of this method is that, in principle, the value function of a suitable optimal control problem is the unique mapping that verifies the Dynamic Programming Principle and therefore, the idea is to find an equivalent formulation of the functional equation in terms of a Partial Differential Equation (PDE), the so-called HJB equation.

Hamilton-Jacobi (HJ) equations are fully nonlinear PDEs normally associated with classical mechanics problems. The HJB equation is a variant of the latter and it arises when a dynamical constraint affecting the velocity of the system is present. This constraint in turn, appears frequently in the form a control variable, an input that allows us to change the output of the system in a well definitioned way.

The HJB equation, as mentioned earlier, can also be considered as a differential expression of the Dynamic Programming Principle. Under rather mild assumptions and when no constraints affect directly the system, this fully nonlinear PDE of the first or second order has been shown to be well-posed in the context of viscosity solutions, which were introduced by Crandall and Lions in the 1980s. From the optimal control point of view, the approach consists in calculating the value function associated with the control problem by solving the HJB equation and then identify an optimal control and the associated optimal trajectory. The method has the great advantage of directly reaching the global optimum of the control problem, which is particularly relevant when the problem is non convex, besides providing a constructive procedure for the synthesis of an optimal control in *feedback form*.

For an optimal control problem with n state variables, the application of the dynamic programming principle leads to a HJB equation over a state space of at least the same dimension. It is clear then how a key point for the applicability of the method, is to have effective tools and appropriate numerical techniques to deal with a complexity that grows exponentially with the dimension of the state space.

Moreover, due to physical or economic limitations, we may be forced to include state constraints in the formulation of the optimal control problem. This fact yields to some technical difficulties, for example, the value function may not be continuous nor real-valued, not even for very regular data. Thus, some additional compatibility assumptions involving the dynamics and the state constraints set are required for the characterization of the value function in terms of the HJB equation. This fact can be explained by the lack of information on the boundary of the state constraints.

List of Symbols

$\mathbb{R}^N$	the euclidean $N-$dimensional space
e_j	$j-$vector of the canonical base of the space $\mathbb{R}^N$
$x \cdot y$ or $\langle x, y \rangle$	the scalar product $\sum_{i=1}^{N} x_i y_i$ of vectors $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$.
$\|x\|$	the euclidean norm of $x \in \mathbb{R}^N$, $\|x\| = (x \cdot x)^{1/2}$
$\mathbb{M}_{m \times n}(\mathbb{R})$	the $m \times n$ matrix with values in $\mathbb{R}$
$B(x_0, r)$	the open ball $\{x \in \mathbb{R} : \|x - x_0\| < r\}$
∂E	the boundary of the set E
$d(x, E)$	the distance from x to E (i.e., $d(x, E) = \inf_{y \in E} \|x - y\|$)
$\mathrm{argmin}_E\, u$	the set of minimum points of $u : E \to \mathbb{R}$
$u \vee v$	the maximum between two functions $\max(u, v)(x)$
$u \wedge v$	the minimum between two functions $\min(u, v)(x)$
$\\|u\\|_\infty$	the supremum norm $\sup_{x \in E} \|u(x)\|$ of a function $u : E \to \mathbb{R}$
ω	a modulus, i.e. a function $\omega : [0, +\infty) \to [0, +\infty)$ continuous, non decreasing, and such that $\omega(0) = 0$
$o(t)$ as $t \to a$	a function u such that $\lim_{t \to a} u(t)/t = 0$
$O(t)$ as $t \to a$	a function u such that there exists a $K \in \mathbb{R}$ such that $\lim_{t \to a} u(t)/t = K$
$\partial_x u$	the usual derivative of the function u with respect to the argument x ($x \in E \subset \mathbb{R}$)
$\nabla_x u(x)$	the usual gradient of the function u at x ($x \in E \subset \mathbb{R}^N$), i.e., $\nabla u(x) = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right)$
Δu	the laplacian of the function u, i.e. $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$
$y_x^u(t)$	the state at time t of a control system
$\mathscr{U}(a, b)$	the set of controls, i.e. (Lebesgue) measurable functions $u : [a, b] \to U$
$C(E)$	the space of continuous functions $u : E \to \mathbb{R}$
$Lip(E)$	the space of Lipschitz continuous functions $u : E \to \mathbb{R}$, i.e. such that for some $L \geq 0$ $\|u(x) - u(y)\| \leq \|x - y\|$ for all $x, y \in E$
$C^{0,\gamma}(E)$	the space of $\gamma-$Hölder continuous functions $u : E \to \mathbb{R}$, $\gamma \in (0, 1)$, i.e. $\sup_{x,y \in E} \frac{\|u(x) - u(y)\|}{\|x - y\|^\gamma} < +\infty$
$C^k(\Omega)$	for $k \geq 1$ and Ω open subset of $\mathbb{R}^N$, the subspace of $C(\Omega)$ of functions with continuous partial derivatives in Ω up toorder k
a.e.	Almost everywhere (not true in a set of measure null)
$\square$	end of a proof

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Chapter 1

Bellman's approach to optimal control and viscosity solutions

In this section we present some definitions and the basic theoretical results that will be useful in the following.

1.1 Preliminaries on Control Systems and Optimization problems

We begin by considering a parametrized dynamical system dwelling on $\mathbb{R}^n$:

$$\begin{cases} \dot{y}(s) = f(s, y(s), u(s)), & \text{for a.e. } s \in (t, T) \\ u(s) \in U, & \text{for all } s \in (t, T) \\ y(t) = x. \end{cases} \tag{1.1}$$

The elements that compose a control system are the following: $T \in \mathbb{R} \cup \{+\infty\}$ is the *final horizon*, $t \in (-\infty, T)$ is the *initial time*, $x \in \mathbb{R}^n$ is the *initial position*, $u : \mathbb{R} \to U$ is the *control function* with values in the *control space* U and $f : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is the *dynamics* mapping. In this chapter we assume that

$$U \subseteq \mathbb{R}^m \text{ is compact and nonempty} \tag{H_U}$$

For sake of simplicity, we introduce the set

$$\mathscr{U}(a, b) = \{u : (a, b) \to U \text{ measurable}\}$$

We recall the classic but fundamental result of existence for systems of ordinary differential equations:

Theorem 1.1.1 (Charathéodory)**.** *Assume that*

i) $f(\cdot, \cdot, \cdot)$ *is continuous,* *ii) there exists a positive constant* $L_f > 0$ *such that*

$$|f(t,x,u) - f(t,y,u)| \leq L_f |x-y|$$

for all $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, *and* $u \in U$; *and*

iii) $f(t,x,u(t))$ *is measurable with respect to* t.

Then,

$$y(s) = x + \int_t^s f(z, y(z), u(z))dz$$

is the unique solution of (1.1).

Under the mild hypotheses of the Charathéodory Theorem (above), given a *measurable* control function $u \in \mathscr{U}(t, +\infty)$, the control system (1.1) admits a *unique* solution which is an *absolutely continuous* arc definitioned on $(t, +\infty)$. To emphasize the dependence upon control and initial data, we reserve the notation $y^u_{t,x}(\cdot)$ for such trajectory, which is called the *state* of the control system. In the case of autonomous systems, we assume $t = 0$ and denote the trajectory by $y^u_x(\cdot)$.

Unless otherwise stated, all along this chapter we are assuming that

$$\begin{cases} (i) & f : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n \text{ is continuous.} \\ (ii) & \exists L_f > 0, \forall x, y \in \mathbb{R}^n,\ \forall t, s \in \mathbb{R},\ \forall u \in U \\ & |f(t,x,u) - f(s,y,u)| \leq L_f(|t-s| + |x-y|). \end{cases} \qquad (H_f)$$

Example 1.1.1 (Linear systems). *Suppose that* $U \subseteq \mathbb{R}^m$ *and let* $A(s) \in \mathbb{M}_{n\times n}(\mathbb{R})$ *and* $B(s) \in \mathbb{M}_{n\times m}(\mathbb{R})$ *for any* $s \in (t,T)$. *A linear control system has the structure*

$$\dot{y}(s) = A(s)y(s) + B(s)u(s) \quad \text{and} \quad u(s) \in U, \quad \text{for a.e. } s \in (t,T).$$

Example 1.1.2 (Control-affine systems). *Let* $f_0, \ldots, f_m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *be given vector fields and write* $u(s) = (u_1(s), \ldots, u_m(s))$. *A dynamical system is called control-affine provided* (1.1) *can be written as*

$$\dot{y}(s) = f_0(s, y(s)) + \sum_{i=1}^m f_i(s, y(s))u_i(s) \quad \text{and} \quad u(s) \in U, \quad \text{for a.e. } s \in (t,T).$$

An optimal control problem with fixed final horizon is an optimization problem that aims at minimizing the following functional

$$\mathscr{U}(t,T) \ni u \mapsto \int_t^T e^{-\lambda s}\ell(s, y^u_{t,x}(s), u(s))ds + e^{-\lambda T}\psi(y^u_{t,x}(T)), \qquad (1.2)$$

where $\lambda \geq 0$ is the *discount factor*, $\ell(\cdot)$ is the *running cost* and $\psi(\cdot)$ is the *final cost*.

In this chapter the running cost is supposed to satisfy:

$$\begin{cases} (i) & \ell : \mathbb{R}\times\mathbb{R}^n\times U \to \mathbb{R} \text{ is continuous.} \\ (ii) & \forall R>0,\ \exists L_f^R>0, \forall x,y\in\mathbb{B}_R,\ \forall t,s\in(-R,R),\ \forall u\in U \\ & \qquad |\ell(t,x,u)-\ell(s,y,u)| \le L_\ell^R(|t-s|+|x-y|). \\ (iii) & \exists c_\ell>0,\ \lambda_\ell\ge 1,\ \forall (x,u)\in\mathbb{R}^n\times U: \\ & \qquad 0\le \ell(x,u)\le c_\ell(1+|x|^{\lambda_\ell}). \end{cases} \tag{H_ℓ}$$

The problem takes different name according to the data. In particular, we are interested in two particular problems: the *Infinite Horizon problem* ($T=+\infty$ and $\psi\equiv 0$) and the *Bolza problem* ($T<+\infty$ and $\lambda=0$).

Example 1.1.3 (Quadratic cost)**.** *Let $Q(s)\in\mathbb{M}_{n\times n}(\mathbb{R})$ and $R(s)\in\mathbb{M}_{m\times m}(\mathbb{R})$ for any $s\in(t,T)$, and $P\in\mathbb{M}_{n\times n}(\mathbb{R})$. A quadratic optimal control problem is of the form: for any $s\in(t,T)$, $y\in\mathbb{R}^n$ and $u\in U\subseteq\mathbb{R}^m$*

$$\ell(s,y,u)=\langle Q(s)y,y\rangle+\langle R(s)u,u\rangle \quad \textit{and} \quad \psi(y)=\langle Py,y\rangle.$$

The value function is the mapping that associates any initial time t and initial position x with the optimal value of the problem (1.2). In the case of the *infinite horizon* we are only considering the autonomous case. Hence the value function

$$v(x)=\inf_{u\in\mathscr{U}(0,+\infty)}\left\{\int_0^\infty e^{-\lambda s}\ell(y_x^u(s),u(s))ds\right\}. \tag{1.3}$$

On the other hand, for the *Bolza problem* the value function is

$$v(t,x)=\inf_{u\in\mathscr{U}(t,T)}\left\{\int_t^T \ell(s,y_{t,x}^u(s),u(s))ds+\psi(y_{t,x}^u(T))\right\} \tag{1.4}$$

and in addition it satisfies the next final condition:

$$v(T,x)=\psi(x), \quad \forall x\in\mathbb{R}^n. \tag{1.5}$$

Furthermore, in the formulation of (1.2) we may also consider that the final horizon is not fixed, which leads to a more general class of optimal control processes. Among these, the most relevant for the exposition is the so-called *Minimum time problem* to reach a given target $\Theta\subseteq\mathbb{R}^n$. In this case we write the value function as $T^\Theta(\cdot)$ and name it the *minimum time function*, which is given by

$$T^\Theta(x)=\inf_{u\in\mathscr{U}(0,+\infty)}\{T\ge 0 \mid\ y_x^u(T)\in\Theta\}.$$

This function satisfies by definition the condition $T^\Theta(x)=0$ at any $x\in\Theta$.

1.2 Pontryagin approach to OC

A classic approach to optimal control systems consist in the characterization of one (or more) optimal control via some optimaility conditions. The main results are due to the work of L. Pontryagin and its group [28].

Considering in its generality the system (1.1), a couple $(x(\cdot), u(\cdot))$ is said to be a *process* and $x(\cdot)$ an *admissible trajectory*, if $u(\cdot)$ is a control and $x(\cdot)$ is an absolutely continuous function that solves (1.1). In several applications, the trajectories are required to satisfy an *endpoint constraint*.

To be specific, one supposes that

$$(x(0), x(T)) \in C, \tag{1.6}$$

for a given closed subset $C \subset \mathbb{R}^n \times \mathbb{R}^n$.

As stated previously, an *optimal control problem* consists in finding a control strategy $u(\cdot)$ such that the associated process $(x(\cdot), u(\cdot))$ satisfies the constraint (1.6) and minimizes a given cost functional. Optimal control problems can be formulated in different ways.

For instance, a special case of *Bolza problem* takes the form

$$\text{minimize}\left\{ \hat{\psi}(x(0), x(T)) + \int_0^T L(t, x(t), u(t))\, dt : \; x(\cdot) \in \mathscr{S}_{[0,T]} \right\}, \tag{1.7}$$

(with respect to what stated previously we fix only $t = 0$) where $\hat{\psi} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a given *endpoint cost function*, $L \colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ a given *running cost function*, and $\mathscr{S}_{[0,T]}$ denotes the set of all *feasible* trajectories, i.e. all absolutely continuous functions $x(\cdot)$, that solve (1.1) and (1.6) for some control $u(\cdot)$. Under suitable assumptions, problem (1.7) can be written in the *Mayer formulation*:

$$\text{minimize}\left\{ \psi(x(0), x(T)) : \; x(\cdot) \in \mathscr{S}_{[0,T]} \right\}, \tag{1.8}$$

for some cost function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, via a simple change of variables. In the sequel, we consider optimal control problems written in Mayer formulation.

In general, two types of minima for problem (1.8) are analyzed: *weak* and *strong,* that we define next. Given the norm $\|\cdot\|_\infty$ in the space of the bounded controls $L^\infty([0,T]; \mathbb{R}^m)$, we say that a process $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a *weak local minimizer* of problem (1.8) if there exists $\varepsilon > 0$ such that, for every feasible process $(x(\cdot), u(\cdot))$ satisfying $\|u(\cdot) - \bar{u}(\cdot)\|_\infty < \varepsilon$, we have $\psi(\bar{x}(0), \bar{x}(T)) \leq \psi(x(0), x(T))$. On the other hand, considering the norm $\|\cdot\|_\infty$ in the space $W^{1,1}([0,T]; \mathbb{R}^n)$, we say that a pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a *strong local minimizer* of (1.8) if there exists $\varepsilon > 0$ such that for every feasible process $(x(\cdot), u(\cdot))$ satisfying $\|x(\cdot) - \bar{x}(\cdot)\|_\infty < \varepsilon$ we have $\psi(\bar{x}(0), \bar{x}(T)) \leq \psi(x(0), x(T))$. For bang-singular solutions in control-affine problems it is more natural to consider the L^1-norm in the control space.

The major tool used in Optimal Control Theory to rule out candidates for extremals of problem (1.8), is the well-celebrated Pontryagin Maximum Principle, which is stated in terms of the *(unmaximized) Hamiltonian function*

$$\begin{array}{rcl} H\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^{n,*}\times\mathbb{R}^m & \to & \mathbb{R} \\ (t,x,p,u) & \mapsto & p^\top f(t,x,p,u). \end{array} \tag{1.9}$$

Sometimes we will also refer to H as the *pre-Hamiltonian function*. We state a simple version of this result right after introducing some useful concepts, whose complete description can be found in Vinter [29] or Aubin and Frankowska [2]. Given a closed set $K\subset\mathbb{R}^k$ and a point $\bar{y}\in K$, the *proximal cone* to K at $\bar{y}$ is given by

$$N_K^P(\bar{y}) := \{\eta\in\mathbb{R}^k : \exists M\geq 0 \text{ such that } \eta^\top(y-\bar{y})\leq M|y-\bar{y}|^2, \text{ for all } y\in K\},$$

and the *limiting normal cone* to K at $\bar{y}$ is defined as

$$N_K(\bar{y}) := \{\eta\in\mathbb{R}^k : \exists y_i \xrightarrow{K} \bar{y} \text{ and } \eta_i\to\eta \text{ such that } \eta_i\in N_K^P(y_i) \text{ for all } i\in\mathbb{N}\}.$$

Here the notation $y_i \xrightarrow{K} \bar{y}$ means that $y_i\in K$ for all $i\in\mathbb{N}$ and $\lim_{i\to+\infty} y_i=\bar{y}$.

The proof of the following statement of the Pontryagin Maximum Principle and of other more general versions can be found in Vinter [29]. We also refer the reader to [28, 7, 12] for further reading on this subject.

Theorem 1.2.1 (Pontryagin Maximum Principle)**.** *Let $(\bar{x},\bar{u})$ be a strong local minimizer for the problem (1.8). Assume that, for some $\delta>0$, the following hypotheses are satisfied.*

(PMP1) *For fixed x,*

$$(t,u)\mapsto f(t,x,u)$$

is Lebesgue $\times$ Borel measurable, i.e. Lebesgue measurable with respect to t and Borel measurable in u. There exists a Lebesgue $\times$ Borel measurable function $k:[0,T]\times\mathbb{R}^m\to\mathbb{R}$, such that $t\mapsto k(t,\bar{u}(t))$ is integrable and, $x\mapsto f(t,x,u)$ is $k(t,u)$-Lipschitz continuous on $\delta\mathbb{B}$ for a.e. $t\in[0,T]$ and $u\in U(t)$, i.e. for a.e. $t\in[0,T]$,

$$|f(t,x,u)-f(t,x',u)|\leq k(t,u)|x-x'|$$

for all $x,x'\in\bar{x}(t)+\delta\mathbb{B}$ and $u\in U(t)$, where $\mathbb{B}$ is the closed unit ball;

(PMP2) *for a.a. $t\in[0,T]$, $u\in U(t)$, the function*

$$x\mapsto f(t,x,u)$$

is continuously differentiable on $\bar{x}(t)+\delta\,\mathrm{Int}\,\mathbb{B}$, where $\mathrm{Int}\,\mathbb{B}$ denotes the interior of the unit ball $\mathbb{B}$;

(PMP3) *the set $\mathrm{Gr}\,U := \{(t,u): t\in[0,T], u\in U(t)\}$ is Lebesgue $\times$ Borel measurable,*

(PMP4) *the cost function ψ is locally continuously differentiable.*

Then there exist $p(\cdot) \in W^{1,1}([0,1];\mathbb{R}^{n,*})$ and $\lambda \geq 0$ such that

(i) $(p,\lambda) \neq (0,0)$;

(ii) $-\dot{p}(t) = D_x H(t,\bar{x}(t),p(t),\bar{u}(t))$ *a.e. on* $[0,T]$;

(iii) $H(t,\bar{x}(t),p(t),\bar{u}(t)) = \max_{u\in U(t)} H(t,\bar{x}(t),p(t),u)$ *a.e. on* $[0,T]$;

(iv) $(p(0),-p(T)) \in \lambda\nabla\psi(\bar{x}(0),\bar{x}(T)) + N_C(\bar{x}(0),\bar{x}(T))$,

here $N_C(\bar{x}(0),\bar{x}(T))$ is the limiting normal cone to C at $(\bar{x}(0),\bar{x}(T))$ as defined above. Moreover, if the problem is such that

$$f(t,x,u) \text{ and } U(t) \text{ are independent of } t,$$

then, in addition to the above conditions, there exists a constant r such that

(v) $H(t,\bar{x}(t),p(t),\bar{u}(t)) = r$, *a.e. on* $[0,T]$.

Example 1 (A Zermelo navigation problem)**.** *Let us consider the following classic navigation problem in a river of non constant stream flow. Let us consider the problem in $\mathbb{R}^2$ and assume that the stream is directed in direction e_1 with speed $v(x_2)$ depending only on x_2 (we can easily assume that for x_2 the speed is zero – we are on the shore of the river – and entering inside the river the stream is more powerful). Our (unitary) control $u \in B(0,1) \subset \mathbb{R}^2$ affects the speed of the trajectory and our aim is to maximize for a time interval $[0,T]$ the traveled distance $x_1(T)$ along the e_1 component. The problem is non trivial since the two opposite choices of direct the control only in the first component (remaining on the shore pointing in the direction to maximize but renouncing to the help given by the stream) and in the second component (pointing only to the second component maximizing in this way the contribution given by the stream speed) are both non optimal.*

Formally the problem is:

$$\begin{cases} \min -x_1(T) \\ \dot{x}_1 = v(x_2) + u_1 \\ \dot{x}_2 = u_2 \\ (x_1(0),x_2(0)) = (0,0) \\ u_1^2 + u_2^2 = 1. \end{cases} \tag{1.10}$$

The Hamiltonian of the problem is clearly

$$H(x,p,u) = p^\top f = [p_1,p_2]\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = p_1(v(x_2)+u_1) + p_2 u_2.$$

The adjoint system is then

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial x_1}H \\ -\frac{\partial}{\partial x_2}H \end{bmatrix} = \begin{bmatrix} 0 \\ -v'(x_2)p_1 \end{bmatrix}$$

with boundary conditions

$$\begin{bmatrix} p_1(T) \\ p_2(T) \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x_1}(T) \\ \frac{\partial \psi}{\partial x_2}(T) \end{bmatrix} = \begin{bmatrix} -1 \\ \dot{0} \end{bmatrix}.$$

We can obtain then that $p(t) = (-1, v'(x_2)t - T)$. *Now the optimality conditions:*

$$\min_{u_1^2+u_2^2} H(x,p,u) = \min_{u_1^2+u_2^2} [p_1, p_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + p_1 v(x_2)$$

since the last term does not participate to the minimization the optimal control is simply $u = -p/|p| = -(p_1/\sqrt{p^1+p^2}, p_2/\sqrt{p^1+p^2})$. *To simplify the successive calculations we assume that* v *is linear and that* $v'(x_2) = 1$. *The optimal control is then*

$$u_1 = \frac{1}{\sqrt{1+(t-T)^2}}, \quad u_2 = \frac{T-t}{\sqrt{1+(t-T)^2}}.$$

1.3 Dynamic Programming Principle and HJB equations

The HJB approach for optimal control problems is based in a functional equation (which is a consequence of the semigroup property of the solutions) known as the *Dynamic Programming Principle*. This equation has different forms based on the issue at hand:

- **Infinite Horizon problem:** for any $x \in \mathbb{R}^n$ and $\tau \in (0, +\infty]$

$$v(x) = \inf_{u \in \mathscr{U}(0,\tau)} \left\{ \int_0^\tau e^{-\lambda s} \ell(y_x^u(s), u(s)) ds + e^{-\lambda \tau} v(y_x^u(\tau)) \right\}. \tag{1.11}$$

- **Bolza problem:** for any $x \in \mathbb{R}^n$ and $\tau \in (t, T)$

$$v(t,x) = \inf_{u \in \mathscr{U}(t,\tau)} \left\{ \int_t^\tau \ell(s, y_{t,x}^u(s), u(s)) ds + v(\tau, y_{t,x}^u(\tau)) \right\}. \tag{1.12}$$

- **Minimum time problem:** for any $x \in \mathbb{R}^n$ and $\tau \in (0, T^\Theta(x))$

$$T^\Theta(x) = \inf_{u \in \mathscr{U}(0,\tau)} \left\{ \tau + T^\Theta(y_x^u(\tau)) \right\}. \tag{1.13}$$

Proposition 1 (DPP for the infinite horizon problem). *Under the assumptions* (H_f) *and* (H_ℓ), *for all* $x \in \mathbb{R}^n$ *and* $\tau > 0$, (1.11) *holds true.*

Proof. Denote by $\bar{v}(x)$ the right-hand side of (1.11) and

$$J_x(\bar{u}) := \int_0^\infty e^{-\lambda s} \ell(y_x^{\bar{u}}(s), \bar{u}(s)) ds.$$

First, we remark that, for any $x \in \mathbb{R}^n$ and $\bar{u} \in \mathscr{U}$,

$$\begin{aligned} J_x(\bar{u}) &= \int_0^\tau e^{-\lambda s}\ell(\bar{y}(s),\bar{u}(s))ds + \int_\tau^\infty e^{-\lambda s}\ell(\bar{y}(s),\bar{u}(s))ds \\ &= \int_0^\tau e^{-\lambda s}\ell(\bar{y}(s),\bar{u}(s))ds + e^{-\lambda\tau}\int_0^\infty e^{-\lambda s}\ell(\bar{y}(s+\tau),\bar{u}(s+\tau))ds \\ &\geq \int_0^\tau e^{-\lambda s}\ell(\bar{y}(s),\bar{u}(s))ds + e^{-\lambda\tau}v(\bar{y}(\tau)) \end{aligned}$$

where $y_x^{\bar{u}}(s)$ is denoted for shortness as $\bar{y}(s)$. Passing to the infimum in the extreme terms of the inequality we get

$$v(x) \geq \bar{v}(x).$$

To prove the opposite inequality, we recall also that $\bar{v}$ is defined as an infimum, so that, for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a control $\bar{u}_\varepsilon$ (and a corresponding trajectory $\bar{y}_\varepsilon$) such that

$$\bar{v}(x) + \varepsilon \geq \int_0^\tau e^{-\lambda s}\ell(\bar{y}_\varepsilon(s),\bar{u}_\varepsilon(s))ds + e^{-\lambda\tau}v(\bar{y}_\varepsilon(\tau)). \tag{1.14}$$

On the other hand, the value function v being defined also as an infimum, for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists a control ∇u_ε such that

$$v(\bar{y}_\varepsilon(\tau)) + \varepsilon \geq J_{\bar{y}_\varepsilon(\tau)}(\tilde{u}_\varepsilon). \tag{1.15}$$

Plugging (1.15) into (1.14), we get

$$\begin{aligned} \bar{v}(x) &\geq \int_0^\tau e^{-\lambda s}\ell(\bar{y}_\varepsilon(s),\bar{u}(s))ds + e^{-\lambda\tau}J_{\bar{y}_\varepsilon(\tau)}(\tilde{u}_\varepsilon) - (1+e^{-\lambda\tau})\varepsilon & (1.16) \\ &\geq J_x(\hat{u}) - (1+e^{-\lambda\tau})\varepsilon & (1.17) \\ &\geq v(x) - (1+e^{-\lambda\tau})\varepsilon, & (1.18) \end{aligned}$$

where $\hat{u}$ is a control defined by

$$\hat{u}(s) = \begin{cases} \bar{u}_\varepsilon(s), & 0 \leq s < \tau, \\ \tilde{u}_\varepsilon(s-\tau), & s \geq \tau. \end{cases}$$

Since ε is arbitrary, (1.16) finally yields $\bar{v}(x) \geq v(x)$. □

Let us assume that the value functions are continuously differentiable functions and that the infimum in the Dynamic Programming Principle is attained. Hence, some standard calculations yield to the following HJB equation:

- **Infinite Horizon problem:**

$$\lambda v(x) + H(x, \nabla v(x)) = 0, \quad x \in \mathbb{R}^n, \tag{1.19}$$

with $H(x,p) := \sup\{-\langle f(x,u), p\rangle - \ell(x,u) \mid u \in U\}$.

- **Bolza problem:**

$$-\partial_t v(t,x) + H(t,x,\nabla_x v(t,x)) = 0, \quad (t,x) \in (-\infty,T)\times\mathbb{R}^n,$$

with $H(t,x,p) := \sup\{-\langle f(t,x,u),p\rangle - \ell(t,x,u) \mid u \in U\}$.

- **Minimum problem:**

$$-1 + H(x,\nabla T^{\Theta}(x)) = 0, \quad x \in \mathrm{int}(\mathrm{dom}\, T^{\Theta}),$$

with $H(x,p) := \sup\{-\langle f(x,u),p\rangle \mid u \in U\}$.

Proof. We report here a short heuristic proof of the first case above. We underline that the proof is valid only for the (simple) case of a smooth solution. For the general proof refer to [4][Prop. III, 2.8]. Let us write the (1.11) for the infinitesimal case $\tau = h$ and let us consider a first order approximation (Euler scheme) for the trajectory $y_x^u(h) = x + hf(0,x,u) + O(h^2)$. We have

$$v(x) = \inf_{u\in\mathscr{U}(0,\tau)} \left\{ \int_0^h e^{-\lambda s}\ell(x+O(s),u(s))ds + e^{-\lambda h} v(x+hf(0,x,u)) \right\}.$$

Considering $e^{-\lambda s} = 1 - \lambda h + O(h^2)$ and a standard first order approximation for the integral we obtain, for an optimal control $u^* \in U$

$$v(x) = h\ell(x,u^*) + (1-\lambda h)v(x+hf(0,x,u^*)) + O(h^2).$$

After dividing for h and interpreting $(v(x) - v(x+hf(x,u^*)))/h \approx -f(x,h)\cdot\nabla_x v(x)$ as directional derivative, we get, for $h \to 0$,

$$\lambda v(x) - f(x,u^*)\cdot\nabla v(x) - \ell(x,u^*) = 0$$

which is the differential form required.

□

However, the value function is rarely differentiable and so solutions to the Hamilton-Jacobi-Bellman equations need to be understood in a weak sense. The most suitable framework to deal with these equations is the *Viscosity Solutions Theory* introduced by Crandall and Lions in 1983 in their famous paper [13].

In the next section we introduce the notion of *viscosity solution* of the HJ equation

$$F(x,v(x),\nabla v(x)) = 0, \quad x \in \Omega \tag{1.20}$$

where Ω is an open domain of $\mathbb{R}^n$ and the Hamiltonian $F = F(x,r,p)$ is a continuous, real valued function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$. Later we will discuss further hypothesis on the Hamiltonian. The notion of viscosity solution, allows us to obtain important existence and uniqueness results for some equations of the form (1.20) and to establish a link with the dynamic programming principle.

Remark 1.3.1. *The equation* (1.20) *can depend on time, describing the evolution of a system. In that case it is definitioned in the space* $(-\infty,T)\times\Omega$ *with* $T\in\mathbb{R}$ *and it is*

$$F(t,x,v(t,x),\partial_t v(t,x),\nabla_x v(t,x)) = 0, \quad t\in(0,T), x\in\Omega \tag{1.21}$$

1.4 Viscosity solutions

We deal with the equation

$$F(x, v(x), \nabla v(x)) = 0, \quad x \in \Omega \tag{HJ}$$

where Ω is an open domain of $\mathbb{R}^n$ and the function $F = F(x, r, p)$ is a real valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ verifying the following requests

- H1 - $F(\cdot,\cdot,\cdot)$ is uniformly continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$,
- H2 - $F(x, u, \cdot)$ is convex in $\mathbb{R}^n$,
- H3 - $F(x,\cdot,\cdot)$ is monotone.

We note that in the various cases presented in the previous section, H1-H3 are naturally verified as a consequence of the regularity assumptions on the data of the optimal control problem associated.

It is well know that equation (HJ) is in general not well-posed in the classical sense. That is, it is possible to show several examples in which no continuously differentiable solution exists. Furthermore, it is possible to construct an infinite number of almost everywhere differentiable solutions. For example, let us consider a simple 1-dimensional Eikonal equation with a Dirichlet boundary condition, that is

$$\begin{cases} |\nabla v(x)| = 1, & x \in (-1, 1) \\ v(x) = 0, & x = \pm 1 \end{cases} \tag{1.22}$$

This equation admits an infinite number of almost everywhere differentiable solutions (see Fig. 1.1). The theory of viscosity solutions was developed in order to overcome these problems. It gives a way to get uniqueness of the solution and in some cases also to select the solution that has a physical interpretation.

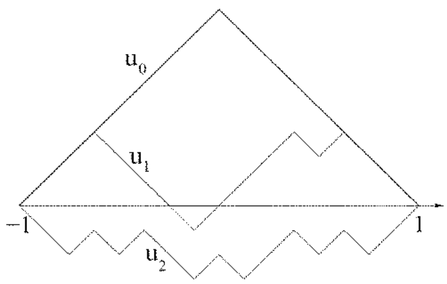

Figure 1.1: Multiple a.e. differentiable solutions of the eikonal equation (1.22).

Definition 1. *A continuous function $v : \Omega \to \mathbb{R}$ is a* viscosity solution *of the equation* (HJ) *if the following conditions are satisfied:*

- *for any test function $\phi \in C^1(\Omega)$, if $x_0 \in \Omega$ is a local maximum point for $v-\phi$, then*

$$F(x_0, v(x_0), \nabla\phi(x_0)) \leq 0 \quad \textit{(viscosity subsolution)}$$

- *for any test function $\phi \in C^1(\Omega)$, if $x_0 \in \Omega$ is a local minimum point for $v-\phi$, then*

$$F(x_0, v(x_0), \nabla\phi(x_0)) \geq 0 \quad \textit{(viscosity supersolution)}$$

Remark 1.4.1. *The notion of viscosity solution can also be extended to the case of lower semicontinuous value functions; for instance if ψ is only lower semicontinuous in the Bolza problem. In this case the solutions are usually called bilateral viscosity solutions; see for instance [4, Chapter 5.5].*

The motivation for the terminology *viscosity solutions* is that this kind of solution can be recovered as the limit function $v = \lim_{\varepsilon\to 0^+} v^\varepsilon$ where $v^\varepsilon \in C^2(\Omega)$ is the classical solution of the parabolic problem

$$-\varepsilon\Delta v^\varepsilon + F(x, v^\varepsilon, \nabla v^\varepsilon) = 0, \quad x \in \Omega. \tag{1.23}$$

Observe that passing to the limit in (1.23) as $\varepsilon \to 0^+$ in order to recover equation (HJ) is not an easy task. This is due to the nonlinearity of the equation and the fact that the required estimates on v^ε, which can be assumed to be smooth, explode as $\varepsilon \to 0^+$ since the regularizing effect of the additive term $-\varepsilon\Delta v^\varepsilon$ becomes weaker and weaker. On the other hand, one does not expect (HJ) to have smooth solutions.

Proof: Viscosity limit. Assume therefore that $v^\varepsilon \in C^2(\mathbb{R}^N)$ converges locally uniformly as $\varepsilon \to 0^+$ to some $v \in C(\mathbb{R}^N)$. Now let ϕ be a C^2 function and x be a strict maximum point for $v-\phi$. By uniform convergence, $v^\varepsilon - \phi$ attains a local maximum at some point x^ε and $x^\varepsilon \to x$ as $\varepsilon \to 0^+$. Hence, by elementary computations,

$$\nabla(v^\varepsilon - \phi)(x^\varepsilon) = 0, \quad -\Delta(v^\varepsilon - \phi) \geq 0.$$

By (1.23), then

$$-\varepsilon\Delta\phi(x^\varepsilon) + F(x, v(x^\varepsilon), \nabla\phi(x^\varepsilon)) \leq 0, \quad x \in \mathbb{R}^N.$$

Since $x^\varepsilon \to 0^+$ as $\varepsilon \to 0^+$, we can pass to the limit in the preceding using the continuity of $\Delta\phi$, $\nabla\phi$ and ϕ and $F(\cdot, v, \cdot)$.

The conclusion is that

$$F(x, v(x), \nabla\phi(x)) \leq 0,$$

therefore v is a viscosity subsolution of (HJ). In a similar way one can show that v is a supersolution in the viscosity sense. □

This method is named *vanishing viscosity*, and it is the original idea behind this notion of solution proposed by Crandall and Lions in [13].

The following proposition explains the local character of the notion of viscosity solution and its consistency with the classical pointwise definition.

Proposition 2. *(a) If $v \in C(\Omega)$ is a viscosity solution of* (HJ) *in Ω, then v is a viscosity solution of* (HJ)*, for any open set $\Omega' \subset \Omega$.*

(b) If $v \in C(\Omega)$ is a classical solution of (HJ)*, that is, u is differentiable at any $x \in \Omega$ and*

$$F(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \Omega, \tag{1.24}$$

then v is a viscosity solution of (HJ).

(c) If $u \in C^1(\Omega)$ is a viscosity solution of (HJ)*, then u is a classical solution of* (HJ).

Proof. (a) If x_0 is a local maximum (on Ω') for $v - \phi$, $\phi \in C^1(\Omega')$, then x_0 is a local maximum (on Ω) for $v - \tilde{\phi}$, for any $\tilde{\phi} \in C^1(\Omega)$ such that $\tilde{\phi} \equiv \phi$ on $\bar{B}(x_0, r)$ for some $r > 0$. By the definition of subsolution

$$0 \geq F(x_0, v(x_0), \nabla\tilde{\phi}(x_0)) = F(x_0, v(x_0), \nabla\phi(x_0))$$

showing that u is a viscosity subsolution also on Ω'. The same argument applies to prove that u is also a supersolution on Ω'.

(b) Take any $\phi \in C^1(\Omega)$. By the differentiability of v, at any local maximum or minimum $x \in \Omega$ of $v - \phi$ we have $\nabla v(x) = \nabla\phi(x)$. Hence (1.24) yields

$$0 = F(x_0, v(x_0), \nabla\phi(x_0)) \leq 0$$

if x_0 is a local maximum for $v - \phi$ and

$$0 = F(x_1, v(x_1), \nabla\phi(x_1)) \geq 0$$

if x_1 is a local minimum for $v - \phi$.

(c) If $u \in C^1(\Omega)$, then $\phi \equiv v$ is a feasible choice in the definition of viscosity solution. With this choice, any $x \in \Omega$ is simultaneously a local maximum and minimum for $v - \phi$. Hence by the definitiontion of sub-supersolution

$$F(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \Omega.$$

□

Remark 1. *Can this notion of weak solution solve the ambiguity in the example* (1.22)*?*

Let us observe that taking any a.e solution u which has a local minimum at x_0 and choose (for example) $\phi = costant$. Clearly, x_0 is a local minimum point for $v - \phi$ so that we should have

$$|\phi_x(x_0)| \geq 1$$

which is false *since $\phi_x \equiv 0$. Then every a.e. solution having a local minimum point in $(1,1)$ cannot be a viscosity super-solution.*

Note that, the same argument will not work for sub-solutions: taking any a.e solution u which has a local maximum at x_0 and choose (for example) $\phi = costant$. Clearly, x_0 is a local maximum point for $v - \phi$ so that we should have

$$|\phi_x(x_0)| \leq 1$$

which is true *since* $\phi_x \equiv 0$.

The only viscosity solution of (1.22) *is then* $v(x) = 1 - |x|$.

Viscosity subsolutions (respectively, supersolutions) are stable with respect to the max (respectively, the min) operator. Introducing, for $u, v \in C(\Omega)$, the notations

$$(u \vee v)(x) = \max\{u(x), v(x)\}, \tag{1.25}$$
$$(u \wedge v)(x) = \min\{u(x), v(x)\}, \tag{1.26}$$

we have the following stability result.

Proposition 3. *The following statements hold true:*

- *(i) Let $u, v \in C(\Omega)$ be viscosity subsolutions of the equation* (HJ)*. Then, $u \vee v$ is a viscosity subsolution.*
- *(ii) Let $u, v \in C(\Omega)$ be viscosity supersolutions of the equation* (HJ)*. Then, $u \wedge v$ is a viscosity supersolution.*

Proof. Let x_0 be a local maximum point for $(u \vee v) - \phi$ where $\phi \in C^1(\Omega)$ is our test function. Without loss of generality, we can assume that $(u \vee v)(x_0) = u(x_0)$. Since x_0 is local maximum point for $u - \phi$, we have

$$F(x_0, u(x_0), \nabla\phi(x_0)) \leq 0,$$

which proves (i). The reverse asserction (ii) can be proven in a similar way. □

An important property which follows from Proposition 3 is that the viscosity solution u can be characterized as the maximal subsolution of the equation, i.e.,

$$u \geq v \quad \text{for any } v \in S, \tag{1.27}$$

where S is the space of subsolutions, i.e.,

$$S = \{v \in C(\Omega) : \text{ and condition (i) of Def. 1 is satisfied}\}.$$

Proposition 4. *Let $u \in C(\Omega)$ be a viscosity subsolution of* (HJ)*, such that $u \geq v$ for any viscosity subsolution $v \in C(\Omega)$. Then, u is a viscosity supersolution and therefore a viscosity solution of* (HJ).

Proof. We will prove the result by contradiction. Assume that

$$d := F(x_0, u(x_0), \nabla\phi(x_0)) < 0$$

for some $\phi \in C^1(\Omega)$ and $x_0 \in \Omega$ such that

$$u(x_0) - \phi(x_0) \leq u(x) - \phi(x), \quad \forall x \in B(x_0, \delta_0) \subset \Omega$$

for some $\delta_0 > 0$. Now, consider the function $w \in C^1(\Omega)$ defined as

$$w(x) := \phi(x) - |x - x_0|^2 + u(x_0) - \phi(x_0) + \frac{1}{2}\delta^2$$

for $0 < \delta < \delta_0$. It is easy to check that, by construction,

$$(u - w)(x_0) < (u - w)(x) \quad \forall x \text{ such that } |x - x_0| = \delta. \tag{1.28}$$

We prove now that, for some sufficiently small δ,

$$F(x, w(x), \nabla w(x)) \leq 0 \quad \forall x \in B(x_0, \delta). \tag{1.29}$$

For this purpose, a local uniform continuity argument shows that, for $0 < \delta < \delta_0$,

$$|\phi(x) - \phi(x_0)| \leq \omega_1(\delta), \tag{1.30}$$

$$|\nabla\phi(x) - 2(x - x_0) - \nabla\phi(x_0)| \leq \omega_2(\delta) + 2\delta \tag{1.31}$$

for any $x \in B(x_0, \delta)$, where the ω_i, $i = 1, 2$, are the moduli of continuity of, respectively, ϕ and $\nabla\phi$. Then, we have

$$|w(x) - u(x_0)| \leq \omega_1(\delta) + \delta^2, \quad x \in B(x_0, \delta).$$

Now, adding and subtracting d

$$F(x, w(x), \nabla w(x)) = d + F(x, w(x), \nabla\phi(x) - 2(x - x_0)) - F(x_0, w(x_0), \nabla\phi(x_0)).$$

Denoting by ω the modulus of continuity of F, we can write

$$F(x, w(x), \nabla w(x)) \leq d + \omega(\delta) + \omega(\omega_1(\delta) + \delta^2) + \omega(\omega_2(\delta) + 2\delta)$$

for all $x \in B(x_0, \delta)$. Since d is negative, the above inequality proves (1.29) for $\delta > 0$ small enough. Let us fix such a δ and set

$$\hat{v}(x) := \begin{cases} u \vee w, & x \in B(x_0, \delta), \\ u, & x \in \Omega \setminus B(x_0, \delta). \end{cases} \tag{1.32}$$

It is easy to check that, by (1.28), $\hat{v} \in C(\Omega)$, so by Propositions 2 and 3 $\hat{v}$ is a subsolution of (HJ). Since $\hat{v}(x_0) > u(x_0)$, the statement is proved. □

We present some comparison results between viscosity sub- and supersolutions. As simple corollary, each comparison result produces a uniqueness theorem for the associated Dirichlet problem. In the following of the section we assume F of the form $F(x,r,q) = ar + H(x,q)$ where the positive constant a (possibly zero) will be specified in each case.

Theorem 1.4.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that v_1, $v_2 \in C(\overline{\Omega})$ are, respectively, viscosity sub- and supersolution of*

$$v(x) + H(x, \nabla v(x)) = 0, \quad x \in \Omega \tag{1.33}$$

and

$$v_1 \leq v_2 \quad \text{on } \partial\Omega. \tag{1.34}$$

Assume also that H satisfies

$$|H(x,p) - H(y,p)| \leq \omega_1(|x-y|(1+|p|)), \tag{1.35}$$

for $x, y \in \Omega$, $p \in \mathbb{R}^n$, where ω_1 is a modulus*, that is $\omega_1 : [0,+\infty) \to [0,+\infty)$ is continuous non decreasing with $\omega_1(0) = 0$. Then $v_1 \leq v_2$ in $\overline{\Omega}$.*

Proof. Define, for $\varepsilon > 0$, a continuous function Φ_ε on $\bar{\Omega} \times \bar{\Omega}$ by setting

$$\Phi_\varepsilon(x,y) = v_1(x) - v_2(y) - \frac{|x-y|^2}{2\varepsilon}$$

and let $(x_\varepsilon, y_\varepsilon)$ be a maximum point for Φ_ε on $\bar{\Omega} \times \bar{\Omega}$. Then, for any $\varepsilon > 0$

$$\max_{x \in \bar{\Omega}} (v_1 - v_2)(x) = \max_{x \in \bar{\Omega}} \Phi_\varepsilon(x,x) \leq \max_{(x,y) \in (\bar{\Omega})^2} \Phi_\varepsilon(x,y) = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon). \tag{1.36}$$

We claim that

$$\liminf \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leq 0 \text{ as } \varepsilon \to 0. \tag{1.37}$$

This, together with (1.36), proves the theorem.

In order to prove (1.37), let us observe first that the inequality

$$\Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$$

amounts at

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \leq v_2(x_\varepsilon) - v_2(y_\varepsilon).$$

This implies

$$|x_\varepsilon - y_\varepsilon| \leq \sqrt{C\varepsilon},$$

where C depends only on the maximum of $|v_2|$ in $\bar{\Omega}$. Therefore

$$|x_\varepsilon - y_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0; \tag{1.38}$$

and by continuity of v_2,

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.39}$$

Now there are two possible cases:

- (i) $(x_{\varepsilon_n}, y_{\varepsilon_n}) \in \partial(\Omega \times \Omega)$ for some sequence $\varepsilon_n \to 0^+$;
- (ii) $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ for all $\varepsilon \in (0, \bar{\varepsilon})$.

In case (i) either $x_{\varepsilon_n} \in \partial\Omega$, and then by (1.34),

$$v_1(x_{\varepsilon_n}) - v_2(y_{\varepsilon_n}) \leq v_2(x_{\varepsilon_n}) - v_2(y_{\varepsilon_n}),$$

or $y_{\varepsilon_n} \in \partial\Omega$ and then

$$v_1(x_{\varepsilon_n}) - v_2(y_{\varepsilon_n}) \leq v_1(x_{\varepsilon_n}) - v_1(y_{\varepsilon_n}).$$

Note that the right-hand sides of both these inequalities tend to 0 as $n \to \infty$ by (1.38) and the uniform continuity of v_1 and v_2. Therefore

$$\Phi_{\varepsilon_n}(x_{\varepsilon_n}, y_{\varepsilon_n}) \leq v_1(x_{\varepsilon_n}) - v_2(y_{\varepsilon_n}) \to 0 \quad \text{as } n \to \infty,$$

and the claim (1.37) is proved in this case.

Assume now that $(x_\varepsilon, y\varepsilon) \in \Omega \times \Omega$ and set

$$\phi_2(x) = v_2(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon}, \quad \phi_1(y) = v_1(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon}.$$

It is immediate to check that $\phi_i \in C^1(\Omega)$ $(i = 1,2)$ and x_ε is a local maximum point for $v_1 - \phi_2$, whereas y_ε is a local minimum point for $v_2 - \phi_1$. Moreover,

$$\nabla\phi_1(y_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = \nabla\phi_2(x_\varepsilon).$$

By the definition of viscosity sub- and supersolution, then,

$$v_1(x_\varepsilon) + H\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \leq 0, \quad v_2(y_\varepsilon) + H\left(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \geq 0.$$

Subtracting the two inequalities above we have

$$v_1(x_\varepsilon) - v_2(y_\varepsilon) \leq \omega_1\left(|x_\varepsilon - y_\varepsilon|\left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right)\right)$$

and even more

$$\Phi_{\varepsilon_n}(x_{\varepsilon_n}, y_{\varepsilon_n}) \leq \omega_1\left(|x_\varepsilon - y_\varepsilon|\left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right)\right).$$

Taking (1.38) and (1.39) into account, (1.37) follows and the proof is complete. □

Remark 2. *If v_1 and v_2 are both viscosity solutions of* (1.45) *with $v_1 = v_2$ on $\partial\Omega$, from the theorem above it follows that $v_1 = v_2$ in $\bar{\Omega}$.*

Theorem 1.4.2. *Assume that $v_1, v_2 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ are, respectively, viscosity sub- and supersolution of*

$$v(x) + H(x, \nabla v(x)) = 0, \quad x \in \mathbb{R}^n. \tag{1.40}$$

Assume also that H satisfies (1.46) *and*

$$|H(x,p) - H(x,q)| \leq \omega_2(|p-q|), \quad \textit{for all } x,p,q \in \mathbb{R}^n. \tag{1.41}$$

where ω_2 is a modulus. *Then $v_1 \leq v_2$ in $\mathbb{R}^n$.*

Remark 3. *Theorem 1.4.2 (for the proof we refer to [4] Chapter II, Theorem 3.5.) can be generalized to cover the case of a general unbounded open set $\Omega \subset \mathbb{R}^n$. Moreover, the assumptions $v_1, v_2 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ can be replaced by $v_1, v_2 \in UC(\mathbb{R}^n)$ (uniformly continuous).*

A comparison result for the more general case

$$H(x, \nabla v(x)) = 0, \quad x \in \Omega \tag{1.42}$$

can be stated if we assume the convexity of H with respect to the p variable. This assumption plays a key role in many theoretical results.

Theorem 1.4.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that $v_1, v_2 \in C(\overline{\Omega})$ are, respectively, viscosity sub- and supersolution of* (1.42) *with $v_1 \leq v_2$ on $\partial\Omega$. Assume also that H satisfies* (1.46) *and the two following conditions*

- *$p \to \bar{H}(x,p)$ is convex on $\mathbb{R}^n$ for each $x \in \Omega$,*
- *there exists $\phi \in C(\overline{\Omega}) \cap C^1(\Omega)$ such that $\phi \leq v_1$ in $\overline{\Omega}$ and $\sup_{x \in B} \bar{H}(x, D\phi(x)) < 0$ for all $B \subset \Omega$.*

Then $v_1 \leq v_2$ in Ω.

The proof of this result can be found in [4] Chapter II, Theorem 5.9.

1.5 Time dependent case

Despite we focused more on the time-independent case, the same definition and results as in the following could be shown in the time-dependent framework of the form (1.21). To see that it is sufficient to make the standard transformation

$$y = (x,t) \in \Omega \times [0,T] \subseteq \mathbb{R}^{n+1}, \quad \tilde{F}(y,r,q) = q_{n+1} + F(x,r,(q_1,...,q_n)) \tag{1.43}$$

where $q = (q_1, ..., q_{n+1}) \in \mathbb{R}^{n+1}$.

For the benefit of the reader here we shortly report the basic results and definitions adapted for the time dependent case.

Consider the time dependant HJ equation

$$\begin{cases} v_t(t,x) + F(x, v(t,x), \nabla_x v(t,x)) = 0 & (t,x) \in (0,T) \times \Omega \\ v(0,x) = v_0(x), & x \in \Omega \\ v(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \end{cases} \tag{1.44}$$

where Ω is an open domain of $\mathbb{R}^n$ and the Hamiltonian $F = F(x,r,p)$ is a continuous, real valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. We define the following notion of weak solution

Definition 2. *A continuous function $v : (0,T) \times \Omega \to \mathbb{R}$ is a* viscosity solution *of the equation* (1.44) *if the following conditions are satisfied:*

- *for any test function $\phi \in C^1((0,T) \times \Omega)$, if $(t_0,x_0) \in (0,T) \times \Omega$ is a local maximum point for $v - \phi$, then*

$$\phi_t(t_0,x_0) + F(x_0, v(x_0), \nabla\phi(x_0)) \leq 0 \quad \textit{(viscosity subsolution)}$$

- *for any test function $\phi \in C^1((0,T) \times \Omega)$, if $(t_0,x_0) \in (0,T) \times \Omega$ is a local minimum point for $v - \phi$, then*

$$\phi_t(t_0,x_0) + F(x_0, v(x_0), \nabla\phi(x_0)) \geq 0 \quad \textit{(viscosity supersolution)}$$

The uniqueness of the solution can be stated through the following theorem

Theorem 1.5.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that $v_1, v_2 \in C([0,T], \overline{\Omega})$ are, respectively, viscosity sub- and supersolution of*

$$v_t(t,x) + \lambda v(t,x) + H(x, \nabla_x v(t,x)) = 0, \quad (t,x) \in (0,T) \times \Omega. \tag{1.45}$$

Assume also that H satisfies

$$|H(x,p) - H(y,p)| \leq \omega_1(|x-y|(1+|p|)), \tag{1.46}$$

for $x,y \in \Omega$, $p \in \mathbb{R}^n$, where ω_1 is a modulus*, that is $\omega_1 : [0,+\infty) \to [0,+\infty)$ is continuous non decreasing with $\omega_1(0) = 0$. Then*

$$\sup_{[0,T]\times\bar{\Omega}} (v_1 - v_2) \leq \sup_{0\times\bar{\Omega}} (v_1 - v_2)^+.$$

The proof of the result above is (with caution in some technical points) the same of Theo. 1.4.1.

1.6 Representation formulae and Legendre transform

In some cases it is possible to derive representation formulae for viscosity solutions. These formulae have a great importance from both the analytical and the numerical points of view and will be derived here in two major cases: linear advection equations and convex HJ equations.

Linear advection equation Let us start by the linear case in which the representation formula can be obtained via the *method of characteristics*.

Theorem 1.6.1. *Let $v : (t_0,T)\times\mathbb{R}^n \to \mathbb{R}$ be a viscosity solution of the initial value problem*

$$\begin{cases} v_t(t,x)+\lambda v(t,x)+f(t,x)\cdot\nabla v(t,x)=g(t,x), & (t,x)\in(t_0,T)\times\mathbb{R}^n,\\ v(t_0,T)=v_0(x), & x\in\mathbb{R}^n.\end{cases} \tag{1.47}$$

Assume that $f:(t_0,T)\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:(t_0,T)\to\mathbb{R}$ are continuous in (t,x) and f is globally Lipschitz continuous with respect to x. Then,

$$v(t,x)=e^{\lambda(t_0-t)}v_0(y(x,t;t_0))+\int_{t_0}^{t}e^{\lambda(s-t)}g(y(x,t;s),s)ds, \tag{1.48}$$

where $y(x,t;s)$ is the position at time s of the solution trajectory passing through x at time t, i.e. solving the Cauchy problem

$$\begin{cases} \frac{d}{ds}y(x,t;s)=f(y(x,t;s),s),\\ y(x,t;t)=x.\end{cases} \tag{1.49}$$

Proof. We give the proof under the additional assumption that $u\in C^1$.

Let (t,x) be fixed, and denote for shortness the solution of (1.49) as $y(s)=y(x,t;s)$. Writing the equation in (1.47) at a point $(s,y(s))$ and multiplying by $e^{\lambda s}$, we have

$$e^{\lambda s}v_s(s,y(s))+\lambda e^{\lambda s}v(s,y(s))+e^{\lambda s}f(s,y(s))\cdot\nabla v(s,y(s))=e^{\lambda s}g(s,y(s)).$$

Since v is differentiable, this may also be rewritten as

$$\frac{d}{ds}\left[e^{\lambda s}v(s,y(s))\right]=e^{\lambda s}g(s,y(s)).$$

Integrating such equation over the interval $[t_0,t]$ we get

$$e^{\lambda t}v(t,y(t))=e^{\lambda t_0}v(t_0,y(t_0))+\int_{t_0}^{t}e^{\lambda s}g(s,y(s))ds.$$

Recalling that $y(t)=y(x,t;t)=x$ and $v(t_0,y(t_0))=u_0(y(t_0))$ and dividing by $e^{\lambda t}$, we get

$$v(t,x)=e^{\lambda(t_0-t)}v_0(y(x,t;t_0))+\int_{t_0}^{t}e^{\lambda(s-t)}g(y(x,t;s),s)ds,$$

which is (1.48). □

We recall that a solution of (1.49) is called *characteristic curve*.

Remark 4 (Stationary case)**.** *Note that if λ is strictly positive, f and g do not depend on t, and the source g is bounded, then $v(t,x)$ has a limit for $t-t_0 \to \infty$. Setting conventionally $t=0$ and letting $t_0 \to -\infty$, we obtain in fact the limit*

$$v(x) = \int_{-\infty}^{0} e^{\lambda s} g(y(x,0,s))ds = \int_{0}^{\infty} e^{\lambda s} g(y(x,0,-s))ds,$$

which is a regime solution for problem (1.47) *or, in other terms, solve the stationary equation*

$$\lambda v(x) + f(x)\cdot\nabla v(x) = g(x)$$

for $x \in \mathbb{R}^n$.

HJ equations Concerning Hamilton Jacobi equations, the representation formula is known as *Hopf-Lax formula* and is typically related to the problem

$$\begin{cases} v_t + H(\nabla u) = 0, & (t,x) \in (0,T)\times\mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.50}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is convex and satisfies the coercivity condition

$$\lim_{|p|\to+\infty} \frac{H(p)}{|p|} = +\infty. \tag{1.51}$$

Such assumption allows us to give the following definition.

Definition 3. *We define the* Legendre-Fenchel conjugate *(or transform) of H for $q \in \mathbb{R}^n$ as*

$$H^*(q) = \sup_{p\in\mathbb{R}^n} \{p\cdot q - H(p)\}. \tag{1.52}$$

Note that being H continuous and also coercive (in the sense of as above), the sup in (1.52) is in fact a maximum. In general, the LegendreFenchel transform may not allow for an explicit computation. A few examples, anyway, can be computed analytically, among which is the quadratic Hamiltonian

$$H_2(p) = \frac{|p|^2}{2},$$

for which an easy computation gives

$$H_2^*(q) = \frac{|q|^2}{2}.$$

The definition of Legendre-Fenchel transform may also work for the noncoercive case, but in this case the conjugate function will not in general be defined everywhere and be bounded. For example, taking

$$H_1(p) = |p|$$

and using the definition (1.52), it is possible to check that

$$H_1^*(p) = \begin{cases} 0 & \text{for } |q| \leq 1, \\ +\infty & \text{elsewhere.} \end{cases} \tag{1.53}$$

The main result of interest here concerns two important properties of the Legendre transform.

Theorem 1.6.2. *Let* (1.51) *be satisfied. Then, the function* H^* *has the following properties:*

(i) $H^* : \mathbb{R}^n \to \mathbb{R}$ *is convex and*

$$\lim_{|p| \to +\infty} \frac{H^*(p)}{|p|} = +\infty;$$

(ii) $H(p) = H^{**}(p)$ *for any* $p \in \mathbb{R}^n$.

Proof. The proof can found in [18][p.122]. □

The above theorem says that by applying the Legendre transform to H twice we obtain back H itself. The definition of Legendre transform is very useful for characterizing the unique solution of (1.50) by means of the so-called *HopfLax representation formula*. The following theorem provide the basic results concerning this characterization.

Theorem 1.6.3. *The function v defined by the following Hopf-Lax formula:*

$$v(t,x) = \inf_{y \in \mathbb{R}^n} \left\{ v_0(y) + tH\left(\frac{x-y}{t}\right) \right\} \tag{1.54}$$

is Lipschitz continuous, is differentiable a.e. in $(0, +\infty) \times \mathbb{R}^n$*, and solves in the viscosity sense the initial value problem* (1.50)*.*

Proof. The proof can found in [18][pp.120-124]. □

1.7 The Eikonal equation

The classical model problem for (1.42) is the Eikonal equation on geometric optics

$$c(x)|\nabla v(x)| = 1, \quad x \in \Omega \tag{1.55}$$

Theorem 1.4.3 applies to the eikonal equation (1.55) whenever $c(x) \in Lip(\Omega)$ and it is strictly positive. In fact the second condition of Theorem 1.4.3 is satisfied by taking $\phi(x) \equiv \min_{\overline{\Omega}} u_1$.

It is easy to prove that the distance function from an arbitrary set $S \subseteq \mathbb{R}^n$, $S \neq \emptyset$ defined by

$$d_S(x) = \text{dist}(x, S) := \inf_{z \in S} |x - z| = \min_{z \in \overline{S}} |x - z| \tag{1.56}$$

is continuous in $\mathbb{R}^n$. Moreover, for smooth ∂S, it satisfies in the classical sense the equation (1.55) in $\mathbb{R}^n \setminus \overline{S}$ for $c(x) \equiv 1$.

For a general set S, it can be shown that the function d_S is the unique *viscosity solution* of

$$|\nabla v(x)| = 1, \quad x \in \mathbb{R}^n \setminus \overline{S} \tag{1.57}$$

Remark 5. *If we consider the eikonal equation in the form $|\nabla v(x)| = g(x)$ where g is a function vanishing at least in a single point in Ω, then the uniqueness result does not hold. This situation is referred to as* degenerate eikonal equation. *It can be proved that in this case many viscosity or even classical solution may appear. Consider for example the equation $|v'| = 2|x|$ for $x \in (-1,1)$ complemented by Dirichlet boundary condition $v = 0$ at $x = \pm 1$. It is easy to see that $v_1(x) = x^2 - 1$ and $v_2(x) = 1 - x^2$ are both classical solutions. The case of degenerate eikonal equations has been studied by Camilli and Siconolfi [9] and numerically by Camilli and Grüne in [8].*

The minimum time problem Let us come back to the minimum time problem to reach a given closed *target* $\Theta \subset \mathbb{R}^n$. Note that a priori nothing implies that the end-point constraint

$$y_x^u(T) \in \Theta$$

will be satisified for any $x \in \mathbb{R}^n$ and $u \in \mathscr{U}(0,T)$. This implies that the minimum time function may not be well defined in some regions of the space, which in mathematical terms means that $T^\Theta(x) = +\infty$.

Definition 4. *The* reachable set *is $\mathscr{R}^\Theta := \{x \in \mathbb{R}^n : T^\Theta(x) < +\infty\}$, i.e. it is the set of starting points from which it is possible to reach the target Θ.*

Note that the reachable set depends on the target, the dynamics and on the set of admissible controls and it is not a datum in our problem. The importance of the reachable set is reflected by the following result

Proposition 5. *If $\mathscr{R}^\Theta \setminus \Theta$ is open and $T^\Theta \in C(\mathscr{R}^\Theta \setminus \Theta)$, then T^Θ is a viscosity solution of*

$$\max_{u \in U} \{-f(x,u) \cdot \nabla T(x)\} - 1 = 0, \quad x \in \mathscr{R}^\Theta \setminus \Theta \tag{1.58}$$

A more detailed proof of the result above can be found in [4] Chapter IV, Proposition 2.3. The uniqueness of the solution can be proved under an additional condition of *small time local controllability* (for further details we refer to [4] Chapter IV, Theorem 2.6). Natural boundary conditions for (1.58) are

$$\begin{cases} T^\Theta(x) = 0 & x \in \partial\Theta \\ \lim\limits_{x \to \partial\mathscr{R}^\Theta} T^\Theta(x) = +\infty. \end{cases} \tag{1.59}$$

In order to archive uniqueness of the viscosity solution of equation (1.58) is useful an exponential transformation named *Kruzkov transform*

$$v(x) := \begin{cases} 1 - e^{-T^\Theta(x)} & \text{if } T(x) < +\infty \\ 1 & \text{if } T(x) = +\infty \end{cases} \tag{1.60}$$

It is easy to check (at least formally) that if T^Θ is a solution of (1.58) then v is a solution of

$$v(x) + \max_{u\in U}\{-f(x,u)\cdot\nabla v(x)\} - 1 = 0, \quad x \in \mathbb{R}^n \setminus \Theta. \tag{1.61}$$

This transformation has many advantages:

- The equation for v has the form (1.45) so that we can apply the uniqueness result already introduced in this chapter.
- v takes value in $[0,1]$ whereas T^Θ is generally unbounded (for example if f vanishes in some points) and this helps in the numerical approximation.
- The domain in which the equation has to be solved is no more unknown.
- One can always reconstruct T^Θ and $\mathscr{R}^\Theta$ from v by the relations

$$T^\Theta(x) = -\ln(1 - v(x)), \quad \mathscr{R}^\Theta = \{x : v(x) < 1\}.$$

Optimal feedback and trajectories Let us consider for simplicity the minimum time problem. As mentioned above, the final goal of every optimal control problem is to find the a control $u^* \in \mathscr{U}(0, T^\Theta(x))$ such that

$$T^\Theta(x) = \inf\{\tau > 0 \mid y_x^{u^*}(\tau) \in \Theta\} \tag{1.62}$$

The next theorem shows how to compute u^* in feedback form, i.e. as a function of the state $y(t)$. This form turns out to be more useful than open-loop optimal control where u^* depends only on time t. In fact, the feedback control leads the state to the target even in presence of perturbations and noise.

Theorem 1.7.1. *Assume that a function $T^\Theta \in C^1(\mathscr{R}^\Theta \setminus \Theta)$ be the unique viscosity solution of* (1.58) *and suppose that the mapping $\kappa(x)$ defined below is continuous*

$$\kappa(x) := \arg\max_{u\in U}\left\{-f(x,u)\cdot\nabla T^\Theta(x)\right\}, \quad x \in \mathscr{R}^\Theta \setminus \Theta. \tag{1.63}$$

Let $y^*(t)$ be the solution of

$$\begin{cases} \dot{y}^*(t) = f(y^*(t), \kappa(y^*(t))), & t > 0 \\ y^*(0) = x \end{cases} \tag{1.64}$$

Then, $u^*(t) = \kappa(y^*(t))$ an optimal control.

The result above is related to some regularity issues. The regularity of the value function in the minimum time case is a delicate issue and it was discussed and studied in several works. A detailed presentation of the problem can be found in the Chapter IV of [4] and in [11, 3]. More recent works are [10], [22], [23].

Chapter 2

Numerical schemes for Hamilton-Jacobi equations

In this chapter we deal with the numerical approximation of Hamilton-Jacobi equations.

2.1 Crandall-Lions and Barles-Souganidis theorems

We collect together these two results which make use of monotonicity as a stability assumption. The Crandall-Lions theorem is inspired by the result of convergence of monotone conservative schemes for conservation laws; therefore it assumes the scheme to have a structure which parallels the structure of conservative schemes. On the contrary, the Barles-Souganidis theorem does not assume any particular structure for the scheme, and is suitable for more general situations (including second-order HJ equations), provided a comparison principle holds for the exact equation. It also requires a more technical definition of consistency.

We present this result referring to the case of one d-space dimensions. With a small abuse of notation, we rewrite

$$v_t + H(\nabla v) = 0 \tag{2.1}$$

as

$$v_t + H(v_{x_1}, v_{x_2}, ..., v_{x_d}) = 0. \tag{2.2}$$

The Crandall-Lions theorem works in the framework of difference schemes, so we assume that the space grid is orthogonal and uniform, Δx_i being the space step along the i-th direction. We define an approximation of the partial derivative u_{x_i} at the point x_j by the right finite difference, that is,

$$\Delta_{i,j}(v) = \frac{v_{j+e_i} - v_j}{\Delta x_i} \quad (i = 1, 2, ..., d).$$

In parallel with the definition of schemes in conservative form for conservation laws, we define here the class of schemes in *differenced form*.

Definition 5. *A scheme S is said to be in differenced form if it has the form*

$$v_j^{n+1} = v_j^n - \Delta t \mathscr{H}(\Delta_{1,j-p}(v^n), \cdots, \Delta_{1,j+q}(v^n); \Delta_{2,j-p}(v^n), \cdots, \Delta_{2,j+q}(v^n)) \quad (2.3)$$

for two multi-indices p and q with positive components and for a Lipschitz continuous function $\mathscr{H}$ (called the numerical Hamiltonian).

In practice, (2.3) defines schemes in which the dependence on v^n appears only through its finite differences, computed on a rectangular stencil of points around x_j. The differenced form of a scheme lends itself to an easier formulation of the consistency condition, which is given in the following.

Definition 6 (Consistency in differenced form)**.** *A scheme in differenced form is consistent if, for any $a, b \in \mathbb{R}$,*

$$\mathscr{H}(a, \cdots, a; b, \cdots, b) = H(a, b).$$

Remark 6. *We point out that the previous definition matches the usual one for schemes in differenced form. To show this fact, we rewrite the scheme in the form*

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \mathscr{H}(\Delta_{1,j-p}(v^n), \cdots, \Delta_{1,j+q}(v^n); \Delta_{2,j-p}(v^n), \cdots, \Delta_{2,j+q}(v^n)) = 0$$

substituting the finite differences with the Tayor expansion of the terms in $\mathscr{H}$ we get the usual notion of consistency.

Note also that, in the nonlinear case, we expect that monotonicity may or may not hold depending on the speed of propagation of the solution, this speed being related to the Lipschitz constant of u_0 (in fact, in general, monotonicity does depend on the speed of propagation, but in the linear case this speed is given and unrelated to u_0).
We have now all elements to state the Crandall-Lions convergence theorem.

Theorem 2.1.1 (Crandall-Lions)**.** *Let $H : \mathbb{R}^d \to \mathbb{R}$ be continuous, let v_0 be bounded and Lipschitz continuous (with Lipschitz constant L) on $\mathbb{R}^2$. Let the scheme* (2.3) *be monotone on $[-(L+1), L+1]$ and consistent for a locally Lipschitz continuous numerical Hamiltonian $\mathscr{H}$. Then there exists a constant C such that, for any $n \leq T/\Delta t$ and $j \in \mathbb{Z}^d$*

$$|v_j^n - v(x_j, n\Delta t)| \leq C(\Delta t)^{\frac{1}{2}}, \quad \forall j, \forall n \leq N \quad (2.4)$$

as $\Delta t \to 0$, $\Delta x_i = \lambda_i \Delta t$ $(i = 1, 2 ... d)$.

Proof. We only prove

$$v(x_j, n\Delta t) - v_j^n \leq C\sqrt{\Delta x},$$

since the reverse inequality is obtained in a very similar way. To avoid confusion in this proof we call the piecewise linear extension of v_j^n in $\mathbb{R}^d$ as $w(t, x)$ so then $v_j^n = w(x_j, n\Delta t)$. Let us define

$$M = \sup_{\mathscr{G}^\Delta} \{v(x, t) - w(x, t)\}$$

in the point $(\widehat{t},\widehat{x})$ and assume that $v(\widehat{t},\widehat{x}) \leq w(\widehat{t},\widehat{x})$ so $M > 0$. The opposite case can be treated similarly.
For every $\varepsilon, \beta, \eta \in (0,1)$ and $\sigma > 0$, we define the auxiliary function

$$\psi(t,s,x,y) := v(x,t) - w(y,s) - \frac{|t-s|^2 + |x-y|^2}{2\varepsilon} - \beta|x|^2 - \frac{\eta}{T-t} - \sigma t,$$

$$\text{for } (x,t),(y,s) \in \mathbb{R}^d \times [0, T = n\Delta t]. \quad (2.5)$$

Using some basic regularity properties we deduce that the function ψ is bounded an then achieves its maximum at some point $(\bar{t},\bar{s},\bar{x},\bar{y})$, i.e.

$$\psi(\bar{t},\bar{s},\bar{x},\bar{y}) \geq \psi(t,s,x,y) \quad \text{for all } (t,x) \in [0,T) \times J,\ (s,y) \in \mathscr{G}^{\Delta}.$$

We denotes by K several positive constant only depending on the Lipchitz constants of u and w and on the final time T.

Step 1. (Basic estimates). The maximum point of ψ satisfy the following estimates:

$$|\bar{x} - \bar{y}| = O(\varepsilon), \quad |\bar{t} - \bar{s}| = O(\varepsilon). \quad (2.6)$$

From $\psi(\bar{t},\bar{s},\bar{x},\bar{y}) \geq \psi(\bar{t},\bar{s},\bar{y},\bar{y})$ we get

$$\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \leq v(\bar{t},\bar{x}) - v(\bar{t},\bar{y}) - \beta|\bar{x}|^2 \leq v(\bar{t},\bar{x}) - v(\bar{t},\bar{y}). \quad (2.7)$$

By the Lipschtz property of v (standard in the case of viscosity solutions), there exists a positive constant K such that

$$\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \leq K|\bar{x}-\bar{y}|, \quad (2.8)$$

then the first estimate of (2.6).

The second bound in (2.6) is deduced from $\psi(\bar{t},\bar{s},\bar{x},\bar{y}) \geq \psi(\bar{s},\bar{s},\bar{y},\bar{y})$ in the same way. From (2.7) it is also direct to obtain the estimate

$$|\bar{x}| \leq O(\varepsilon). \quad (2.9)$$

Step 2. (Viscosity inequalities).

We claim that for σ large enough, the supremum of ψ is achieved for $\bar{t} = 0$ or $\bar{s} = 0$. We prove the assertion by contradiction. Suppose $\bar{t} > 0$ and $\bar{s} > 0$.

Using the fact that $(t,x) \to \psi(t,\bar{s},x,\bar{y})$ has a maximum in $(\bar{x},\bar{t})$ and that u is a sub solution we get

$$\frac{\bar{t}-\bar{s}}{\varepsilon} + \frac{\eta}{(T-\bar{t})^2} + \sigma + H(\varphi_x(\bar{x},\bar{t})) \leq 0 \quad (2.10)$$

where $\varphi(x,t) = \frac{|t-s|^2+|x-y|^2}{2\varepsilon} + \beta|x|^2 + \frac{\eta}{T-t} + \sigma t$.

Since $\hat{s} > 0$ we know that $\psi(\bar{t},\bar{s},\bar{x},\bar{y}) \geq \psi(\bar{t},\bar{s}-\Delta t,\bar{x},y)$ for a generic y and, by defining $\varphi(s,y) = -\frac{|\bar{t}-s|^2+|\bar{x}-y|^2}{2\varepsilon}$, it implies

$$w(\bar{s}-\Delta t, y) \geq \varphi(\bar{s}-\Delta t, y) + w(\bar{s},\bar{y}) - \varphi(\bar{s},\bar{y}).$$

We apply the numerical scheme (2.3) to both terms of the inequality setting $y = \bar{y}$ and by monotonicity, we get

$$w(\bar{s},\bar{y}) = S[\hat{w}(\bar{s}-\Delta t)](\bar{y}) \geq S[\hat{\varphi}(\bar{s}-\Delta t)](\bar{y}) + w(\bar{s},\bar{y}) - \varphi(\bar{s},\bar{y}).$$

Simplifying $w(\bar{s},\bar{y})$, adding and subtracting $\varphi(\bar{s}-\Delta t,\bar{y})$ and dividing by Δt we get

$$\varphi_s(\bar{s},\bar{y}) + \frac{\varphi(\bar{s}-\Delta t,\bar{y}) - S[\hat{\varphi}(\bar{s}-\Delta t)](\bar{y})}{\Delta t} \geq O(\Delta t). \tag{2.11}$$

We subtract (2.11) to (2.10) and using the consistency result we have

$$\frac{\eta}{(T-\bar{t})^2} + \sigma + H\left(\frac{d(\bar{x},\bar{y})}{\varepsilon} + 2\beta|\bar{x}|\right) - H\left(\frac{d(\bar{x},\bar{y})}{\varepsilon}\right) \leq K(\Delta t + \varepsilon).$$

Using the modulus of continuity of H

$$\sigma \leq -\frac{\eta}{(T-\bar{t})^2} + \omega(\varepsilon + 2\beta|\bar{x}|) + K(\Delta t + \varepsilon) =: \sigma^*. \tag{2.12}$$

In this case we obtain a restriction on σ and then a contradiction with the absurdum hypothesis.

Step 3. (Conclusion). If $\bar{t} = 0$ we have

$$\begin{aligned} M_{\varepsilon,\beta,\eta,\sigma} &\leq v_0(\bar{x}) - w(\bar{s},\bar{y}) - \frac{|\bar{x}-\bar{y}|^2+\bar{s}^2}{2\varepsilon} - \beta|\bar{x}|^2 - \frac{\eta}{T} \\ &\leq v_0(\bar{x}) - v_0(\bar{y}) + C\bar{s} - \frac{\bar{s}^2}{2\varepsilon} \leq L_u|\bar{x}-\bar{y}| + \sup_{r>0}\left(Cr - \frac{r^2}{2\varepsilon}\right) = O(\varepsilon) \end{aligned}$$

A similar argument applies if $\bar{s} = 0$. Finally, for a $\sigma > \sigma^* + O(\beta)$, we have that for all $(t,x) \in [0,T/2] \times \mathbb{R}^d$

$$w(t,x) - v(t,x) \leq \beta|x|^2 + \frac{2\eta}{T-t} + \sigma t + O(\varepsilon)$$

replacing σ by $2\sigma^*$ and optimizing the parameters ($\varepsilon = \sqrt{\Delta t}$) we obtain the desired result for $\eta,\beta \to 0$.

□

The Barles-Souganidis theorem While it still requires monotonicity, the Barles-Souganidis convergence theory [5] gives a more abstract and general framework for convergence of schemes, including the possibility of treating second-order, degenerate, and singular equations. Roughly speaking, this theory states that any monotone, stable, and consistent scheme converges to the exact solution, provided there exists a comparison principle for the limiting equation. The Cauchy problem under consideration is

$$\begin{cases} v_t + H(x, v, \nabla v, \nabla^2 v) = 0 & \text{on } (0,T) \times \mathbb{R}^d, \\ v = v_0 & \text{on } \{0\} \times \mathbb{R}^d. \end{cases} \tag{2.13}$$

The function $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times M^d \to$ (where M^d is the space of $d \times d$ symmetric matrices) is a continuous Hamiltonian, possibly depending on the Hessian matrix $\nabla^2 v$ of second derivatives, and is assumed to be elliptic, i.e.,

$$H(x, w, p, A) \leq H(x, w, p, B)$$

for all $w \in \mathbb{R}$, $x, p \in \mathbb{R}^d$, $A, B \in M^d$ such that $A \geq B$, that is, such that $A - B$ is a positive semidefinite matrix. In particular, this form includes the first-order HJ equation (2.24).

We will assume that a comparison principle holds true for (2.13); i.e., we assume that if v and w are, respectively, a super- and a subsolution of (2.13) on $\mathbb{R}^d \times (0,T) \to \mathbb{R}$, and if $v(\cdot, 0) \leq w(\cdot, 0)$, then $v \leq w$.

Let us consider a scheme in the general form (**??**). First, we require the property of *invariance with respect to the addition of constants* i.e.

$$S(V + c) = S(V) + c, \quad \text{for any vector } V.$$

Then, a *generalized consistency condition* is assumed as follows.

Definition 7. *Let $\Delta_m = (\Delta x_m, \Delta t_m)$ be a generic sequence of discretization parameters, and let (x_{j_m}, t_{n_m}) be a generic sequence of nodes in the space-time grid such that, for $m \to \infty$,*

$$(\Delta x_m, \Delta t_m) \to 0 \text{ and } (x_{j_m}, t_{n_m}) \to (x, t).$$

Let $\phi \in C^\infty(\mathbb{R}^d \times (0, T])$. Then, the scheme S is said to be consistent *if*

$$\liminf_{m \to \infty} \frac{\phi(x_{j_m}, t_{n_m}) - S(\Delta_m; \Phi(t_{n_m - 1}))}{\Delta t_m} \geq \phi_t(x,t) + \underline{H}(x, \phi(x,t), \nabla \phi(x,t), \nabla^2 \phi(x,t)),$$

$$\limsup_{m \to \infty} \frac{\phi(x_{j_m}, t_{n_m}) - S(\Delta_m; \Phi(t_{n_m - 1}))}{\Delta t_m} \leq \phi_t(x,t) + \overline{H}(x, \phi(x,t), \nabla \phi(x,t), \nabla^2 \phi(x,t)).$$

Here, the index of the sequence is m, j_m , and n_m denoting the corresponding indices of a node with respect to the m-th space-time grid, and we recall that by Φ or $\Phi(t)$ we denote the vector of node values for, respectively, $\phi(x)$ and $\phi(x,t)$. Moreover, $\overline{H}$ and $\underline{H}$ denote here lower and upper semicontinuous envelopes of H:

$$\underline{H}(x, \phi(x,t), \nabla \phi(x,t), \nabla^2 \phi(x,t)) = \liminf_{(y,s) \to (x,t)} H(y, \phi(y,s), \nabla \phi(y,s), \nabla^2 \phi(y,s))$$

$$\overline{H}(x,\phi(x,t),\nabla\phi(x,t),\nabla^2\phi(x,t)) = \limsup_{(y,s)\to(x,t)} H(y,\phi(y,s),\nabla\phi(y,s),\nabla^2\phi(y,s))$$

Note that if H is continuous (and this is the standard case considered in the rest of the notes, the lim inf and the lim sup must coincide, and the definition reduces to the usual definition of consistency. The standard definition of monotonicity is also replaced by a generalized monotonicity assumption stated as follows.

Definition 8. *Let $(\Delta x_m, \Delta t_m)$ and (x_{j_m}, t_{n_m}) be generic sequences satisfying*

$$(\Delta x_m, \Delta t_m) \to 0 \text{ and } (x_{j_m}, t_{n_m}) \to (x,t).$$

Then, the scheme S is said to be monotone (in the generalized sense) if it satisfies the following conditions:

$$\text{if } v_{j_m} \le \phi_{j_m}, \text{ then } S_{j_m}(\Delta_m; V) \le S_{j_m}(\Delta_m; \Phi) + o(\Delta t_m); \tag{2.14}$$

$$\text{if } v_{j_m} \ge \phi_{j_m}, \text{ then } S_{j_m}(\Delta_m; \Phi) \le S_{j_m}(\Delta_m; V) + o(\Delta t_m); \tag{2.15}$$

for any smooth function $\phi(x)$.

Also in this case, we have that if a scheme is monotone in the usual form (4.13), then it also satisfies (2.14)–(2.15). Now, consider a numerical solution V_n (with v_j^n and its piecewise constant (in time) interpolation $v^\Delta t$) defined as

$$v^{\Delta t}(x,t) = \begin{cases} \mathbb{I}[V^n](x) & \text{if } t \in [t_n, t_{n+1}), \\ v_0(x) & \text{if } t \in [0, \Delta t). \end{cases}$$

Here, $\mathbb{I}[V^n]$ is assumed to be a general interpolation operator

$$\mathbb{I}[V^n](x) = \sum_{x_l \in \mathscr{S}(x)} \psi_l(x) v_l^n, \tag{2.16}$$

where $\{\psi_l\}$ is a basis of cardinal functions in $\mathbb{R}^d$ (which in particular satisfy the property $\sum_l \psi_l(x) = 1$, and $\mathscr{S}(x)$ is the stencil of nodes involved for interpolating at the point x. We assume here that it is contained in a ball of radius $O(\Delta x)$ around x and refer the reader to the section on the semilagrangian scheme for a detailed treatment of the various interpolation techniques. The interpolation operator also has to verify a relaxed monotonicity property:

$$\text{if } v_j \le \phi_j, \text{ for any } j \text{ that } x_j \in \mathscr{S}(x), \text{ then } \mathbb{I}[V](x) \le \mathbb{I}[\Phi](x) + o(\Delta t_m); \tag{2.17}$$

$$\text{if } v_j \ge \phi_j, \text{ for any } j \text{ that } x_j \in \mathscr{S}(x), \text{ then } \mathbb{I}[\Phi](x) \le \mathbb{I}[V](x) + o(\Delta t_m); \tag{2.18}$$

where V and ϕ denote vectors of node values of, respectively, a generic numerical solution and a smooth function $\phi(x)$. Moreover, $\mathbb{I}[\cdot]$ satisfies

$$|\mathbb{I}[\Phi](x) - \Phi(x)| = o(\Delta t) \tag{2.19}$$

Note that, once Δt and Δx are related to one another, bounds (2.17)–(2.18) (which are usually written in terms of the space discretization parameter) may also be understood in terms of Δt.

We can now state the extended version of the convergence result given in [5].

Theorem 2.1.2. *Assume: invariance with respect to the addition of constants, consistency and monotonicity in the weak sense stated above. Let $v(x,t)$ be the unique viscosity solution of* (2.13)*. Then, $v^{\Delta t}(x,t) \to v(x,t)$ locally uniformly on $\mathbb{R}^d \times [0,T]$ as $\Delta \to 0$.*

Proof. Let the bounded functions $\overline{v}$, $\underline{v}$ be defined by

$$\overline{v}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ \Delta t\to 0}} v^{\Delta t}(y,s), \quad \underline{v}(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ \Delta t\to 0}} v^{\Delta t}(y,s),$$

We claim that $\overline{v}(x,t)$, $\underline{v}(x,t)$ are, respectively, a sub- and a supersolution of (2.13). Assume for the moment that the claim is true; then by the comparison principle $\overline{v}(x,t) \le \underline{v}(x,t)$ on $\mathbb{R}^d \times (0,T]$. Since the opposite inequality is obvious by the definition of $\overline{v}(x,t)$ and $\underline{v}(x,t)$, we have

$$v = \overline{v} = \underline{v}$$

and v is the unique continuous viscosity solution of (2.13). This fact together with (2.19) also implies the locally uniform convergence of $v^{\Delta t}$ to v.

Let us prove the previous claim. Let (x,t) be a local maximum of $\overline{v} - \phi$ on $\mathbb{R}^d \times (0,T]$ for some $\phi \in C^\infty(\mathbb{R}^d \times (0,T])$. Without any loss of generality, we may assume that (x,t) is a strict global maximum for $v - \phi$ and that $v(x,t) = \phi(x,t)$. Then, by a standard result from viscosity theory, there exist two sequences $\Delta t_m \in \mathbb{R}^+$ and $(y_m, \tau_m) \in \mathbb{R}^d \times [0,T]$, which are global maximum points for $v_m^{\Delta t} - \phi$, and as $m \to \infty$,

$$\Delta t_m \to 0, \quad (y_m, \tau_m) \to (x,t), \quad v^{\Delta t_m}(y_m, \tau_m) \to \overline{v}(x,t).$$

Then, for any x and t we have

$$v^{\Delta t_m}(x,t) \le \phi(x,t) + \xi_m \tag{2.20}$$

with $\xi_m = (v^{\Delta t_m} - \phi(y_m, \tau_m)$ (note that $v(x,t) = \phi(x,t)$, and hence $\xi_m \to 0$).

Since, in general, (y_m, τ_m) is not a grid point, we need to reconstruct the value at- tained by $v^{\Delta t_m}$ at such points. By the definition of $v^{\Delta t_m}$, there exists a t_{n_m} such that $\tau_m \in [t_{n_m}, t_{n_{m+1}})$ and $v^{\Delta t_m}(y_m, \tau_m) = v^{\Delta t_m}(y_m, t_{n_m})$. Furthermore, by the definition of $\mathbb{I}[\cdot]$ in (2.16), there exists a set of nodes $\mathscr{S}(y_m)$ such that

$$\mathbb{I}[V^{n_m}] = \sum_{x_j \in \mathscr{S}(y_m)} \psi_j(y_m) v_j^{n_m}.$$

Next, we apply (2.20) at $t = t_{n_m - 1}$, $x = x_j \in \mathscr{S}(y_m)$ and deduce, from the invariance with respect to the addition of constants and the monotonicity property, that

$$S_j(\Delta_m; V^{n_m - 1}) \le S_j(\Delta_m; \Phi(t_{n_m - 1})) + \xi_m + o(\Delta_m)$$

Recalling that the left-hand side is nothing but $v_j^{n_m}$, we have

$$v_j^{n_m} \leq S_j(\Delta_m; \Phi(t_{n_m-1})) + \xi_m + o(\Delta t_m),$$

which yields, applying (2.16), (2.17),

$$v^{\Delta t_m}(y_m, \tau_m) \leq \sum_{x_j \in \mathscr{S}(y_m)} \psi_j(y_m) S_j(\Delta_m; \Phi(t_{n_m-1})) + \xi_m + o(\Delta t_m).$$

Now, by the definition of ${}_m$, we get

$$\phi(y_m, \tau_m) \leq \sum_{x_j \in \mathscr{S}(y_m)} \psi_j(y_m) S_j(\Delta_m; \Phi(t_{n_m-1})) + o(\Delta t_m). \tag{2.21}$$

We claim now that $\phi(y_m, \tau_m) = \phi(y_m, t_{n_m}) + O(\Delta t_m^2)$. In fact, either $\tau_m = t_{n_m}$ (and the claim obviously holds), or ${}_m \in (t_{n_m-1}, t_{n_m})$. In the latter case, since $(v^{\Delta t_m} - \phi)(y_m, \cdot)$ has a maximum in ${}_m$ and $v^{\Delta t_m}$ is constant in (t_{n_m-1}, t_{n_m}), then $\phi_t(y_m, \tau_m) = 0$ and we have $\phi(y_m, \tau_m) = \phi(y_m, t_{n_m}) + O(\Delta t_m^2)$.

Using the previous claim in (2.21), we have

$$\phi_j(y_m, t_{n_m}) \leq \sum_{x_j \in \mathscr{S}(y_m)} \psi_j(y_m) S_j(\Delta_m; \Phi(t_{n_m-1})) + o(\Delta t_m). \tag{2.22}$$

and by (2.19)

$$\phi(y_m, t_{n_m}) = \mathbb{I}[\Phi(t_{n_m})](y_m) + o(\Delta t_m) = \sum_{x_j \in \mathscr{S}(y_m)} \psi_j(y_m) \phi(x_j, t_{n_m}) + o(\Delta t_m) \tag{2.23}$$

Now, (2.22) and (2.23) imply

$$\liminf_{m \to \infty} \sum_{x_j \in \mathscr{S}(y_m)} \psi_j(y_m) \frac{\phi(x_j, t_{n_m}) - S(\Delta_m; \Phi(t_{n_m-1}))}{\Delta t_m} + o(1) \leq 0.$$

Finally, by the consistency property, we obtain the desired result:

$$\phi_t(x,t) + H(x, \phi(x,t), \nabla\phi(x,t), \nabla^2\phi(x,t)) \leq 0.$$

The proof that $\underline{v}$ is a supersolution follows the same arguments, except for unsing the second inequality in the consistency definition. We leave this adaptation to the reader. □

2.2 An upwind scheme

We consider the general case

$$v_t + H(v_x) = 0 \tag{2.24}$$

where we will make the standing assumption that H is convex and that there exists $_0 \in \mathbb{R}$ such that

$$\begin{cases} H'(\alpha) \leq 0, & \text{if } \alpha \leq \alpha_0 \\ H'(\alpha) \geq 0, & \text{if } \alpha \geq \alpha_0 \end{cases} \tag{2.25}$$

We also define

$$M_H(L) = \max_{p\in[L,L]} |H'(p)|$$

which corresponds to the maximum speed of propagation of a solution with Lipschitz constant L.

In adapting the upwind scheme to the nonlinear case, it should be taken into consideration that the speed of propagation of the solution is $H'(v_x)$ While it is perfectly clear how to construct an upwind scheme for a speed of constant sign, care should be taken at points where the speed changes sign, in order to obtain a monotone scheme.

The construction outlined will follow the guidelines of [14], in which monotone schemes for HJ equations are derived from monotone schemes for conservation laws, and the theory is carried out accordingly. The differenced form (i.e. depending on finite differences) of the upwind scheme is

$$v_j^{n+1} = v_j^n - \Delta t \mathscr{H}(\Delta_{j-1}v^n, \Delta_i v^n) \tag{2.26}$$

where the numerical Hamiltonian $\mathscr{H}$ is defined by

$$\mathscr{H}(\alpha,\beta) = \begin{cases} H(\alpha) & \text{if } \alpha,\beta \geq \alpha_0 \\ H(\beta) + H(\beta) - H(\alpha_0) & \text{if } \alpha \geq \alpha_0, \beta \leq \alpha_0 \\ H(\alpha_0) & \text{if } \alpha \leq \alpha_0, \beta \geq \alpha_0 \\ H(\beta) & \text{if } \alpha,\beta \leq \alpha_0 \end{cases} \tag{2.27}$$

Note that the situation in which speed changes sign is subject to a different handling, depending on the fact that characteristics converge or diverge.

Consistency. Since the scheme is in differenced form, it actually suffices to apply Definition 6. If $\alpha = \beta$, then the numerical hamiltonian (2.27) satisfies

$$\mathscr{H}(\alpha,\alpha) = H(\alpha)$$

and the consistency condition is satisfied. Note that, in (2.27), the second and third cases occur only if $\alpha = \beta = \alpha_0$.

Stability. By construction, schemes in differenced form (in particular, upwind and semilagrangian in what follows) are necessarily invariant for the addition of constants. Therefore, the main stability issues in this context will be CFL condition and monotonicity.

CFL condition. Since the maximum speed of propagation of the solution is $M_H(L)$, the condition which keeps characteristics within the numerical domain of dependence is CFL condition

$$M_H(L)\frac{\Delta t}{\Delta x} \le 1.$$

In this case, this restriction is really only necessary - as we will see, monotonicity requires a stronger condition.

Monotonicity. As first, we write the partial derivative of the j-th component of the scheme as

$$\frac{\partial}{\partial v_i} S_j(\Delta; v) = \delta_{i,j} - \Delta t \left[\frac{\partial \mathscr{H}}{\partial \alpha} \frac{\partial \Delta_{j-1}(v)}{\partial v_i} + \frac{\partial \mathscr{H}}{\partial \beta} \frac{\partial \Delta_j(v)}{\partial v_i} \right],$$

where α and β are the dummy variables used in the definition (2.27) and $\delta_{i,j}$ is the Kronecker symbol. It is clear that

$$\frac{\partial \Delta_{j-1}}{\partial v_i} = \begin{cases} \frac{1}{\Delta x} & \text{if } i = j, \\ -\frac{1}{\Delta x} & \text{if } i = j-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

$$\frac{\partial \Delta_j}{\partial v_i} = \begin{cases} \frac{1}{\Delta x} & \text{if } i = j+1, \\ -\frac{1}{\Delta x} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

so that, substituting above, we obtain the more explicit form

$$\frac{\partial}{\partial v_i} S_j(\Delta; v) = \begin{cases} -\Delta t \left[\frac{\partial \mathscr{H}}{\partial \alpha} \frac{\partial \Delta_j(v)}{\partial v_{j-1}} \right] & \text{if } i = j-1, \\ 1 - \Delta t \left[\frac{\partial \mathscr{H}}{\partial \alpha} \frac{\partial \Delta_j(v)}{\partial v_j} + \frac{\partial \mathscr{H}}{\partial \beta} \frac{\partial \Delta_j(v)}{\partial v_i} \right] & \text{if } i = j, \\ -\Delta t \left[\frac{\partial \mathscr{H}}{\partial \beta} \frac{\partial \Delta_j(v)}{\partial v_{j+1}} \right] & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.30}$$

By the definition of $\mathscr{H}$ we have

$$\frac{\partial \mathscr{H}}{\partial \alpha}(\alpha, \beta) = \begin{cases} H'(\alpha) \ge 0 & \text{if } \alpha \ge \alpha_0, \\ 0 & otherwise. \end{cases}$$

$$\frac{\partial \mathscr{H}}{\partial \beta}(\alpha, \beta) = \begin{cases} H'(\beta) \le 0 & \text{if } \beta \le \alpha_0, \\ 0 & otherwise. \end{cases}$$

Looking at the signs of the various terms, it is apparent that

$$\frac{\partial}{\partial v_i} S_j(\Delta; V) \ge 0 \quad (i \ne j),$$

whereas, for $i = j$ we have

$$\left| \frac{\partial \mathscr{H}}{\partial \alpha} \frac{\partial \Delta_j(v)}{\partial v_j} + \frac{\partial \mathscr{H}}{\partial \beta} \frac{\partial \Delta_j(v)}{\partial v_i} \right| \le \frac{2M_{H'}(L_v)}{\Delta x},$$

where L_v denotes the Lipschitz constant of the sequence v. Therefore, using this in (2.30), we obtain that the scheme is monotone if

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{2M_{H'}(L_v)}.$$

Note that, in contrast to the linear case, this condition is more stringent then the CFL condition.

Finally, we give the convergence result, which follows from consistency, monotonicity, and Crandall-Lions Theorem.

Theorem 2.2.1 (Crandall-Lions for Upwind)**.** *Let H satisfy the basic assumptions, let $v_0 \in W^{1,\infty}(\mathbb{R})$, let v be the solution of* (2.24) *with L as its Lipschitz constant, and let v_j^n be defined by* (2.26) *with $v_j^0 = v_0(x_j)$. Then, for any $j \in \mathbb{Z}$ and $n \in [1, T/\Delta t]$,*

$$|v_j^n - v(x_j, n\Delta t)| \leq C(\Delta t)^{\frac{1}{2}}, \quad \forall j, \forall n \leq N \tag{2.31}$$

as $\Delta \to 0$, with $M_{H'}(L+1)\Delta t \leq \Delta x$.

2.3 The Lax Friedrichs scheme

In treating the LF scheme, we will follow again the guidelines of [14]. Rather than using more general forms of the scheme, we will restrict ourselves here to the particular form that directly generalizes the linear case.

The simplest way to recast the LF scheme for the HJ equation is to define it in the form

$$v_j^{n+1} = \frac{v_{j-1}^n + v_{j+1}^n}{2} - H(\Delta_j^c(v^n)) \tag{2.32}$$

where $\Delta^c(v)$ is the centered difference at x_j defined by

$$\Delta_j^c(v^n) = \frac{v_{j+1}^n + v_{j-1}^n}{2\Delta x} = \frac{\Delta_{j-1}(v^n) + \Delta_j(v^n)}{2}.$$

This definition of the LF scheme completely parallels the linear case and is also suitable to be treated in the framework of the Crandall-Lions theorem. In fact, once we recall that

$$\frac{v_{j-1}^n + v_{j+1}^n}{2} = v_j^n + \frac{\Delta x}{2}\left(\Delta_j(v^n) - \Delta_{j-1}(v^n)\right),$$

(2.32) can be written in the differenced form

$$v_j^{n+1} = v_j^n - \Delta t \mathscr{H}(\Delta_{j-1}(v^n), \Delta_j(v^n))$$

by setting

$$\mathscr{H}(\alpha, \beta) = H\left(\frac{\alpha + \beta}{2}\right) - \frac{\Delta x}{\Delta t}(\beta - \alpha).$$

Note that, as for the advection equation, no special care is necessary to determine the direction of propagation for the solution (that is, to compare α and β with $_0$), since the stencil (i.e. nodes of influence) is symmetric.

Consistency. The LF scheme (2.32) satisfies the consistency condition Def. 6, and in fact

$$\mathscr{H}(a,a) = H((a+a)/2) = H(a).$$

Consistency is therefore satisfied.

Stability. We examine again the issues of CFL condition and monotonicity, which in this case give the same restriction on the discretization steps.
CFL condition. Taking into account that the maximum speed of propagation is $M_{H'}(L)$, the CFL condition reads as

$$\frac{M_{H'}(L_v)\Delta t}{\Delta x} \le 1,$$

as for the upwind scheme. In this case, this condition is necessary and sufficient, since it also ensures monotonicity (as we will soon show).
Monotonicity. In examining monotonicity, it is convenient to refer to the LF scheme in the form (2.32). Clearly, the j-th component $S_j(\Delta; v)$ depends only on the values $v_{j\pm1}$, so that

$$\frac{\partial}{\partial v_i} S_j(\Delta; v) = 0 \quad i \ne j \pm 1.$$

On the other hand, if $i = j \pm 1$, we have

$$\frac{\partial}{\partial v_{j\pm1}} S(\Delta; v) = \frac{1}{2} - \Delta t H'(\Delta_j^c(v)) \frac{\partial \Delta_j^c(v)}{\partial v_{j\pm1}} = \frac{1}{2} \mp \frac{\Delta t}{2\Delta x} H'(\Delta_j^c(v)),$$

where we have used the fact that

$$\frac{\partial \Delta_j^c(v)}{\partial v_{j\pm1}} = \pm \frac{1}{2\Delta x}.$$

Therefore, if L_v is the Lipschitz constant of the sequence V, the scheme is monotone, provided

$$\frac{\Delta t}{\Delta x} \le \frac{1}{M_{H'(L_v)}}.$$

Again, the convergence result is obtained from consistency and monotonicity applying Crandall-Lions Theorem.

Theorem 2.3.1 (Crandall-Lions for Lax-Friedrich)**.** *Let H satisfy the basic assumptions, let $v_0 \in W^{1,\infty}(\mathbb{R})$, let v be the solution of* (2.24) *with L as its Lipschitz constant, and let v_j^n be defined by* (2.32) *with $v_j^0 = v_0(x_j)$. Then, for any $j \in \mathbb{Z}$ and $n \in [1, T/\Delta t]$,*

$$|v_j^n - v(x_j, n\Delta t)| \le C(\Delta t)^{\frac{1}{2}}, \quad \forall j, \forall n \le N \tag{2.33}$$

as $\Delta \to 0$, with $M_{H'}(L+1)\Delta t \le \Delta x$.

2.4 Multiple space dimensions

We briefly turn to the n-dimensional problem:

$$v_t + H(\nabla v) = H(v_{x_1},...,v_{x_d}) = 0, \quad \mathbb{R}^d \times [0,T].$$

Most of the work of extending one-dimensional schemes for HJ equations to the multidimensional case follows the same principles of the linear case. In particular, schemes in differenced form have the general structure (2.3), so that, for example, the two-dimensional version of the LF scheme reads

$$v_{i,j}^{n+1} = \frac{1}{4}\left(v_{i-1,j}^n + v_{i+1,j}^n + v_{i,j-1}^n + v_{i,j+1}^n\right) + \Delta t H\left(\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}, \frac{v_{i,j+1} - v_{i,j+1}}{2\Delta x}\right),$$

and both consistency and monotonicity may be proved by the very same arguments used for the one-dimensional case.

2.5 Semi-Lagrangian approximation for Hamilton-Jacobi equations

It is possible to build some numerical schemes by the discretization of the dynamical programming principle associated to the problem. As a model we will consider the infinite horizon problem. In the current section we introduce a scheme for the stationary case in semi-Lagrangian form. In such an approach the numerical approximation is based on a time-discretization of the original control problem via a discrete version of the Dynamical Programming Principle. Then, the functional equation for the time-discrete problem is "projected" on a grid to derive a finite dimensional fixed point problem. We also show how to obtain the same numerical scheme by a direct discretization of the directional derivatives in the continuous equation. Note that the scheme we study is different to that obtained by Finite Difference approximation. In particular, our scheme has a built-in up-wind correction.

Semi-discrete scheme. The aim of this section is to build a numerical scheme for equation (1.19). In order to do this, we first make a discretization of the autonomous version of the original control problem (1.1) introducing a time step $h = \Delta t > 0$.

We obtain a discrete dynamical system associated to (1.1) just using any one-step scheme for the Cauchy problem. A well known example is the explicit Euler scheme which corresponds to the following discrete dynamical system

$$\begin{cases} y_{n+1} = y_n + hf(y_n,u_n), \quad n = 1,2,... \\ y_0 = x \end{cases} \tag{2.34}$$

where $y_n = y(t_n)$ and $t_n = nh$. We will denote by $y_x(n;\{u_n\})$ the state at time nh of the discrete time trajectory verifying (2.34). We also replace the cost functional

(1.2) by its discretization by a quadrature formula (e.g. the rectangle rule). In this way we get a new control problem in discrete time. The value function v_h for this problem (the analogous of (1.3)) satisfies the following proposition

Proposition 6 (Discrete Dynamical Programming Principle)**.** *We assume that*

$$\exists M > 0 : |\ell(x,u)| \leq M \quad \textit{for all } x \in \mathbb{R}^n, u \in U \tag{2.35}$$

then v_h satisfies

$$v_h(x) = \min_{u \in U}\{(1-\lambda h)v_h(x+hf(x,u)) + \ell(x,u)\}, \quad x \in \mathbb{R}^n. \tag{2.36}$$

This characterization leads us to an approximation scheme, at this time, discrete only on the temporal variable.

Under the usual assumptions of regularity on f and ℓ (Lipschitz continuity, boundedness on uniform norm) and for $\lambda > L_f$ as in (H_f), the family of functions v_h is equibounded and equicontinuous, then, by the Ascoli-Arzelá theorem we can pass to the limit and prove that it converges locally uniformly to v, value function of the continuous problem, for h going to 0. Moreover, the following estimate holds (cf. i.e. [16])

$$||v - v_h||_\infty \leq Ch^{\frac{1}{2}}. \tag{2.37}$$

Fully discrete scheme. In order to compute an approximate value function and solve (2.36) we have to make a further step: a discretization in space. We need to project equation (2.36) on a finite grid. First of all, we restrict our problem to a compact subdomain $\Omega \subset \mathbb{R}^n$ such that, for h sufficiently small

$$x + hf(x,u) \in \overline{\Omega} \quad \forall x \in \overline{\Omega} \quad \forall u \in U. \tag{2.38}$$

We build a regular triangulation of Ω denoting by X the set of its nodes x_i, $i \in I := \{1,...,N\}$ and by S the set of simplices S_j, $j \in J := \{1,...,L\}$. Let us denote by k the size of the mesh i.e. $k = \Delta x := \max_j\{diam(S_j)\}$. Note that one can always decide to build a structured grid (e.g. uniform rectangular meshes) for Ω as it is usual for Finite Difference scheme, although for dynamic programming/semi-Lagrangian scheme is not an obligation. Main advantage of using structured grid is that one can easly find the simplex containing the point $x_i + hf(x_i,a)$ for every node x_i and every control $a \in A$ and make interpolations.

Now we can define the fully discrete scheme simply writing (2.36) at every node of the grid. We look for a solution of

$$v_h^k(x_i) = \min_{u \in U}\{(1-\lambda h)I[v_h^k](x_i + hf(x_i,u)) + h\ell(x_i,u)\}, \quad i = 1,...N \tag{2.39}$$

$$I[v_h^k](x) = \sum_j \lambda_j(x)v_h^k(x_j), \quad 0 \leq \lambda_j(x) \leq 1, \quad \sum_j \lambda_j(x) = 1 \quad x \in \Omega.$$

in the space of piecewise linear functions on Ω. Let us make a number of remarks on the scheme above:

1. The function u is extended on the whole space Ω in a unique way by linear interpolation, i.e. as a convex combination of the values of $v_h^k(x_i)$, $i \in I$. It should be noted that one can choose any interpolation operator. A study of various results of convergence under various interpolation operators are contained in [19].

2. The existence of (at least) one control u^* giving the minimum in (2.39) relies on the continuity of the data and on the compactness of the set of controls.

3. By construction, u belongs to the set

$$W^k := \{w : Q \to [0,1] \text{ such that } w \in C(Q), Dw = costant \text{ in } S_j, j \in J\} \tag{2.40}$$

of the piecewise linear functions.

We map all the values at the nodes onto a N-dimensional vector $V = (V_1, ..., V_N)$ so that we can rewrite (2.39) in a fixed point form

$$V = G(V) \tag{2.41}$$

where $G : \mathbb{R}^N \times \mathbb{R}^N$ is defined componentwise as follows

$$[G(V)]_i := \min_{u \in U} \left[\{(1 - \lambda h) \sum_j \lambda_j (x_i + hf(x_i, u))V_j\} + h\ell(x_i, u) \right]_i \tag{2.42}$$

The proofs of the following results are rather direct with the use of the Banach's fixed point theorem.

Theorem 2.5.1. *For a $\lambda > 0$ and a h small enough to verify $|1 - \lambda h| < 1$, the operator G defined in* (2.42) *has the following properties:*

- *G is monotone, i.e. $U \leq V$ implies $G(U) \leq G(V)$;*
- *G is a contraction mapping in the uniform norm $||W||_\infty = \max_{i \in I} |W_i|$, $\beta \in (0, 1)$*
$$||G(U) - G(V)||_\infty \leq \beta ||U - V||_\infty$$

Proposition 7. *The scheme* (2.39) *has a unique solution in W^k. Moreover, the solution can be approximated by the fixed point sequence*

$$V^{(n+1)} = G(V^{(n)}) \tag{2.43}$$

starting from the initial guess $V^{(0)} \in \mathbb{R}^N$.

There is a global estimate for the numerical solution ([4] Appendix A, Theorem 1.3., see also [15, 5]). Other more recent results are [21, 26]

Theorem 2.5.2. *Let v and v_h^k be the solutions of* (1.19) *and* (2.39)*. Assume the Lipschitz continuity and the boundness of f and ℓ, moreover assume condition* (2.38) *and that $\lambda > L_f$, said L_f, L_ℓ Lipschitz constant of the function f and ℓ, then*

$$||v - v_h^k||_\infty \leq Ch^{\frac{1}{2}} + \frac{L_\ell}{\lambda(\lambda - L_f)} \frac{k}{h}. \tag{2.44}$$

2.6 An application: Solving labyrinths

We propose to use our results on HJ equations on discontinuous data to solve a labyrinth. We propose two different approaches. In the first we can think about a labyrinth as a minimum time problem with constraints, that are the walls. In this case, from the fact that the dynamics is isotropic, the Soner's condition is verified, so we could deal also to this problem with the classical theory of HJ with constraint. This is an alternative approach.

We consider the labyrinth $I(x)$ as a digital image with $I(x) = 0$ if x is on a wall, $I(x) = 0.5$ if x is on the target, $I(x) = 1$ otherwise. We propose to solve the labyrinth shown in Figure 2.1 where the gray square is the target.

We solve the eikonal equation

$$|Du(x)| = f(x) \quad x \in \Omega \tag{2.45}$$

with the discontinuous running cost

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } I(x) = 1 \\ M & \text{if } I(x) = 0. \end{cases} \tag{2.46}$$

We are in the Hypothesis of Chapter 1 so we use the numerical schemes proposed in that Chapter 2. We obtain the value function shown in Figure 2.2. We have chosen $dx = dt = 0.0078$, $M = 10^{10}$.

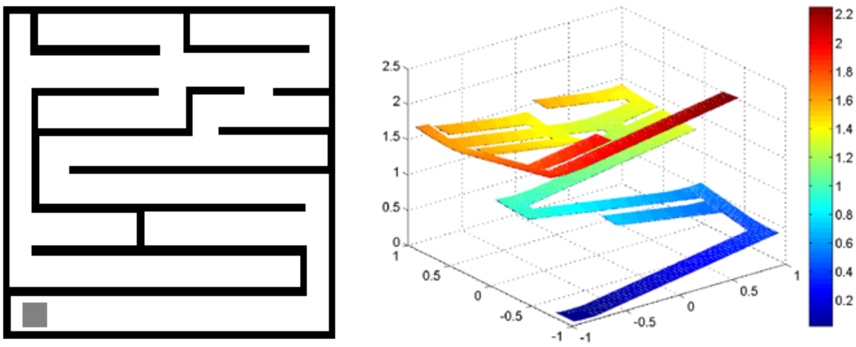

Figure 2.1: A labyrinth as a digital image.

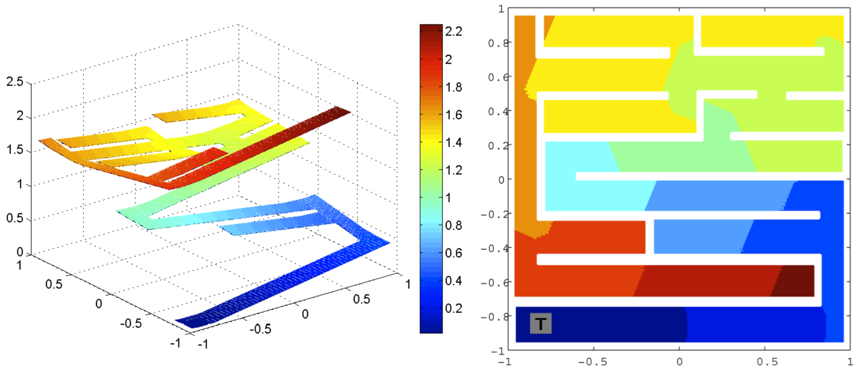

Figure 2.2: Mesh and level sets of the value function for the labyrinth problem.

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