EE2 Mathematics

Solutions to Example Sheet 2: Functions of a complex variable

1) To verify that the following satisfy the Cauchy-Riemann equations $u_x = v_y \quad v_x = -u_y$:

a) $u_x = 1 \quad v_y = 1$; $u_y = v_x = 0$. $\quad \therefore$ CR equations satisfied.

b) $u = e^x \cos y \;\Rightarrow\; u_x = e^x \cos y, \quad u_y = -e^x \sin y$.
$v = e^x \sin y \;\Rightarrow\; v_y = e^x \cos y, \quad v_x = e^x \sin y$. $\quad \therefore$ CR equations satisfied.

c) $u = x^3 - 3xy^2 \;\Rightarrow\; u_x = 3x^2 - 3y^2, \quad u_y = -6xy$.
$v = 3x^2 y - y^3 \;\Rightarrow\; v_x = 6xy, \quad v_y = 3x^2 - 3y^2$. $\quad \therefore$ CR equations satisfied.

2a) With $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ we have $u_x = 3x^2 - 3y^2 + 6x$ and $u_y = -6xy - 6y$. Therefore $u_{xx} = 6x + 6$ and $u_y = -6x - 6$ and so $u_{xx} + u_{yy} = 0$. Because u satisfies Laplace's equation, there exists a conjugate function $v(x, y)$ that satisfies the CR equations: $u_x = v_y, \quad v_x = -u_y$. To find v we integrate these

$$v_y = u_x = 3x^2 - 3y^2 + 6x \quad \Rightarrow \quad v = \int (3x^2 - 3y^2 + 6x)\, dy + A(x)$$
$$v_x = -u_y = 6xy + 6y \quad \Rightarrow \quad v = \int (6xy + 6y)\, dx + B(y)$$

where $A(x)$ and $B(y)$ are arbitrary functions of x and y respectively. The solution(s) for v must be the same from each equation; together we find that $v = 3x^2 y - y^3 + 6xy + c$ where $A = c$ and $B = c - y^3$ with c as an arbitrary constant. In combination $f(z) = u + iv = z^3 + 3z^2$ + const.

2b) $u = xy$ we have $u_x = y$ and $u_y = x$. Therefore $u_{xx} = 0$ and $u_{yy} = 0$ and so $u_{xx} + u_{yy} = 0$. Because u satisfies Laplace's equation, there exists a conjugate function $v(x, y)$ that satisfies the CR equations: $u_x = v_y, \quad v_x = -u_y$. To find v we integrate these

$$v_y = u_x = y \quad \Rightarrow \quad v = \int y\, dy + A(x)$$
$$v_x = -u_y = -x \quad \Rightarrow \quad v = -\int x\, dx + B(y)$$

Together we find that $v = \frac{1}{2}(y^2 - x^2) + c$ where $A(x) = -\frac{1}{2}x^2 + c$ and $B(y) = \frac{1}{2}y^2 + c$. In combination we find that $f(z) = u + iv = -\frac{1}{2}iz^2$+const.

3) To show that the function $v(x, y) = e^x\,(y \cos y + x \sin y)$ satisfies Laplace's equation:

$$v_x = e^x\,(\sin y + y \cos y + x \sin y) \quad \Rightarrow \quad v_{xx} = e^x\,(y \cos y + 2 \sin y + x \sin y)$$
$$v_y = e^x\,(\cos y - y \sin y + x \cos y) \quad \Rightarrow \quad v_{yy} = -e^x\,(2 \sin y + y \cos y + x \sin y)$$

Thus Laplace's equation $v_{xx} + v_{yy} = 0$ is satisfied and we can find a conjugate function u:

$$u_y = -v_x \quad \Rightarrow \quad u = -\int e^x\,(\sin y + y \cos y + x \sin y)\; dy + A(x)$$
$$u_x = v_y \quad \Rightarrow \quad u = \int e^x\,(\cos y - y \sin y + x \cos y)\; dx + B(y)$$

The (partial) integrations are messy but give

$$u = e^x \left(x \cos y - y \sin y \right) + C$$

where $A = B = C =$ const. Using $\int y \cos y dy = \cos y + y \sin y$ and $\int x exp(x) dx = (x-1) exp(x)$. For $f(z) = u + iv$ together we have

$$\begin{aligned} f(z) &= e^x \left(x \cos y - y \sin y + ix \sin y + iy \cos y \right) + c \\ &= e^x z \left(\cos y + i \sin y \right) + c \\ &= e^{x+iy} z + c \\ &= e^z z + c \end{aligned}$$

having used $e^{iy} = \cos y + i \sin y$.

4) The mapping $w = \frac{1}{z-1}$ from the z-plane to the w-plane can be written as

$$w = u + iv = \frac{1}{x - 1 + iy} = \frac{(x-1) - iy}{(x-1)^2 + y^2}$$

$$u = \frac{x-1}{(x-1)^2 + y^2} \qquad v = -\frac{y}{(x-1)^2 + y^2} \qquad \Rightarrow \qquad u^2 + v^2 = \frac{1}{(x-1)^2 + y^2}$$

a) Then the circle $(x-1)^2 + y^2 = 4$ maps to $u^2 + v^2 = \frac{1}{4}$, which is a circle in the w-plane, of radius $\frac{1}{2}$ centred at $(0, 0)$.

b) The line $x = 0$ in the z-plane gives values of u, v

$$u = -\frac{1}{1 + y^2} \qquad v = -\frac{y}{1 + y^2} \qquad \Rightarrow \qquad u^2 + v^2 = \frac{1}{1 + y^2}$$

Hence $u^2 + v^2 = -u$ which, on completing the square, becomes $(u + \frac{1}{2})^2 + v^2 = \frac{1}{4}$. This is a circle in the w-plane, of radius $\frac{1}{2}$ centred at $(-\frac{1}{2}, 0)$.

5) a) For fixed points of $w = \frac{4z-2}{z+1} = z$ solve $z(z+1) = 4z - 2$. Roots occur at $z = 1$ and $z = 2$.

b) For $w = u + iv = \frac{4z-2}{z+1}$ we have

$$u + iv = \frac{4z - 2}{z + 1} = \frac{4x - 2 + 4iy}{x + 1 + iy}$$

Thus solving for u and v through rationalisation of the denominator

$$u = \frac{4(x^2 + y^2) + 2(x-1)}{(x+1)^2 + y^2} \qquad v = \frac{6y}{(x+1)^2 + y^2} \qquad \Rightarrow \qquad (u-1)^2 + v^2 = \frac{9[x^2 + y^2 - 1]^2 + 36y^2}{[(x+1)^2 + y^2]^2}$$

(i) When $x = 0$ in the z-plane then this reduces to $(u-1)^2 + v^2 = 9$. This is a circle in the w-plane of radius 3 centred at $(1, 0)$.

(ii) For the circle $|z| = 1$ in the z-plane we have $x^2 + y^2 = 1$ which means that

$$u = \frac{2x + 2}{2x + 2} = 1 \qquad v = \frac{6y}{2x + 2}$$

Hence in the w-plane we have the vertical line $u = 1$ for all values of v.