A decomposition technique for pursuit evasion games with many pursuers

Adriano Festa and Richard B. Vinter

***Abstract*— Memory storage constraints impose ultimate limits on the complexity of differential games for which optimal strategies can be computed via direct solution of the associated Hamilton-Jacobi-Isaacs equations. It is of interest therefore to explore whether, for certain specially structured differential games of interest, it is possible to decompose the original problem into a family of simpler differential games. In this paper we exhibit a class of single evader-multiple pursuers games for which a reduction in complexity of this nature is possible. The target set is expressed as a union of smaller, sub-target sets. The individual differential games are obtained by substituting a sub-target set in place of the original target and are simpler because of geometric features of the dynamics and constraints. We give conditions under which the value function of the original problem can be characterized as the lower envelope of the value functions for the simpler problems and show how optimal strategies can be constructed from those for the simpler problems. The methodology is illustrated by several examples.**

I. Introduction

Interest in Pursuit-Evasion differential games (*PE games*) involving several agents dates from the 1960's, and is documented in the classical differential games literature, which includes the books by Isaacs [11], Pontryagin [15], Friedman [8], Krasovskii and Subbotin [14]. Constructing optimal strategies, finding the value of the game, deriving optimality conditions for the trajectories and establishing conditions for solvability of the game are typical objectives.

Problems with several agents, though they can be considered as special cases of the general framework for PE games, has been addressed separately by a number of authors. Pshenichnii [16] provided necessary and sufficient optimality conditions for problems involving many pursuers having equal speeds. Ivanov and Ledyaev [12] subsequently studied optimal pursuit problems in higher dimensional state spaces, with several pursuers and geometrical constraints. Through the study of an auxiliary problem, related to the interaction between one pursuer and the evader and using a Lyapunov function, they obtained sufficient conditions of optimality. Chodun [4] and more recently Ibragimov [10] used the same approach for problems of one pursuer and one evader with simple dynamic constraints. Stipanovic et al. [17] constructed approximations to optimal strategies, based on a Lyapunov-type analysis. Differential games, including multi-agent PE games, have been applied in mathematical economics, where games are typically constructed to model the relations between agents [13], [6], in robotics, where often the emphasis is on real time solutions and efficient computational methods [9], [18], and other areas.

Our proposed approach is to decompose the original game into a family of simpler, lower-dimensional games. The original target set is expressed as a union of smaller target subsets. The individual differential games in the family result from replacing the original target set by each of the target subsets. Special geometric features of the dynamic constraints and constraints can be exploited to reduce the complexity of the new problems generated in this way. Using verification techniques originally due to Isaacs [5], [2], properties of viscosity solutions and nonsmooth analysis [1], [3], we provide a lower-envelope characterization of the value function, and construct optimal strategies from those for the simpler problems, c.f. [7].

II. The Hamilton-Jacobi-Isaacs approach to pursuit-evasion games

The state y of a dynamic system, partitioned as n-vector components $y = (y_1, \ldots, y_m)$, is governed by the equations

$$\begin{aligned} y_1'(t) &= -g_1(y(t))a_1(t) + h(y(t))b(t) + l_1(y(t)) \\ &\vdots \\ y_m'(t) &= -g_m(y(t))a_m(t) \\ &\qquad\qquad + h(y(t))b(t) + l_m(y(t)) \ . \end{aligned} \tag{1}$$

in which $g_i(.), h(.) : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ and $l_i(.) : \mathbb{R}^n \to \mathbb{R}^n$ are given functions.

For each i we interpret the state component y_i to be the distance of the evader from the i'th pursuer. The dimension of the state vector y is $N = n \times m$. The N vector $a = (a_1, \ldots, a_m)$ comprises the n-vector pursuer controls and b is the n-vector evader control.

A. Festa is with Imperial College of London, EEE Department, London SW7 2AZ, UK. a.festa@imperial.ac.uk

R. Vinter is with Imperial College of London, EEE Department, London SW7 2AZ, UK. r.vinter@imperial.ac.uk

Pursuer and evader controls a and b take values in

$$A = \mathcal{S}_n(0,\rho_a) \times \ldots \times \mathcal{S}_n(0,\rho_a),\ B = \mathcal{S}_n(0,\rho_b), \tag{2}$$

for some given numbers $\rho_a,\ \rho_b > 0$. Here, $\mathcal{S}_n(0,\rho)$ denotes the closed ball of radius ρ in n-space. Define the control sets as

$$\mathcal{A} := \{ \text{ meas. functions } a : [0,+\infty) \to A\},$$
$$\mathcal{B} := \{ \text{ meas. functions } b : [0,+\infty) \to B\}.$$

It is assumed that

(H): $g_i(.), h(.), l_i(.),\ i = 1,\ldots m+1$ are Lipschitz continuous, and

$$g_i(x)\rho_a - h(x)\rho_b - |l_i(x)| > 0, \quad \forall x \in \mathbb{R}^N, \quad \forall i.$$

Note that the maximum allowable magnitude of the velocity of the i'th pursuer depends on the states of all the pursuers and the evader, via $g_i(.)$, $h(.)$ and $l_i(.)$.

Take the target set to be

$$\mathcal{T} = \{(x_1, x_2, ..., x_m) \in \mathbb{R}^N : \min_{i\in\{1,2\ldots m\}} |x_i| \le r\},$$

in which $r \ge 0$ is a specified number. Denote by $y_x(.) = y_x(., a, b)$ the solution of (1), for given initial state x and controls $a(.) \in \mathcal{A}$ and $b(.) \in \mathcal{B}$. Define the hitting time for given x, a and b to be

$$t_x(a,b) := \begin{cases} \min\{t : y_x(t;a,b) \in \mathcal{T}\} \\ +\infty \quad \text{if } y_x(t;a,b) \notin \mathcal{T}\ \forall t. \end{cases}$$

The player with control a (comprising the pursuers) seeks to minimize the hitting time, while the evader player, with control b, seeks to maximize it.

Remark 2.1: To avoid numerical difficulties associated with an unbounded value function, we replace the hitting time t_x in the cost by its *Kruzkov transform* [1]

$$\psi(t_x(a,b)) = \begin{cases} 1 - e^{-t_x} & \text{if } t_x(a,b) < +\infty \\ 1 & \text{if } t_x(a,b) = +\infty\ . \end{cases}$$

The transformed cost becomes

$$J(x,a,b) = \psi(t_x(a,b)) = \int_0^{t_x} e^{-s} ds.$$

The transformation modifies the value function but not optimal strategies, by monotonicity.

Adopting the Elliot/Kalton 'non-anticiptive controls' framework [5], we define the class of control strategies for the a and b players to be:

$$\Gamma := \{\alpha : \mathcal{B} \to \mathcal{A} : t > 0, b(s) = \tilde{b}(s) \text{ for all } s \le t \text{ implies } \alpha[b](s) = \alpha[\tilde{b}](s) \text{ for all } s \le t\}\ .$$
$$\Delta := \{\beta : \mathcal{A} \to \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \text{ for all } s \le t \text{ implies } \beta[a](s) = \beta[\tilde{a}](s) \text{ for all } s \le t\}\ .$$

The upper and lower values of the game are

$$u^+(x) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} J(x,a,\beta[a])$$
$$u^-(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} J(x,\alpha[b],b)\ .$$

Under the stated hypotheses the upper and lower values coincide, for arbitrary initial state x. The common value defines the value function $u(.)$ for the game. Thus $u(x) = u^+(x) = u^-(x)$ for all x.

It can be shown that, under the state assumptions, the value function is the unique uniformly continuous function $u(.) : \mathbb{R}^N \to \mathbb{R}$, which vanishes on $\mathcal{T}$ and which is a viscosity solution on $\mathbb{R}^N \backslash \mathcal{T}$ of the Hamilton-Jacobi-Isaacs (HJI) equation

$$\begin{cases} u(x) + H(x, Du(x)) = 0 & x \in \mathbb{R}^N \setminus \mathcal{T} \\ u(x) = 0 & x \in \partial\mathcal{T} \end{cases}$$

where The Hamiltonian $H(.,.)$ is

$$H(x, p = (p_1,\ldots,p_m)) = \rho_a \sum_{i=1} g_i(x)|p_i| - \rho_b|\sum_{i=1} h(x)p_i| - \sum_{i=1} l_i(x)\cdot p_i - 1\ .$$

The value function is locally Lipschitz continuous.

Solving this equation yields the optimal strategy of each player for initial state x_0 as $a(t) = S(y_{x_0}(t))$ and $b(t) = W(y_{x_0}(t))$, where

$$S(x) \in \operatorname*{argmax}_{a\in A} \min_{b\in B} \{(g(x)a - h(x)b - l(x))\cdot Du(x)\}$$

$$W(x) \in \operatorname*{argmin}_{b\in B} \max_{a\in A} \{(g(x)a - h(x)b - l(x))\cdot Du(x)\}.$$

Example 1. As a simple illustrative example consider a P-E game with two pursuers p_1, p_2 and one evader e, whose positions evolve in 1D space. If the speeds of evader and the two pursuers are bounded by $\frac{1}{2}$, 1 and $\frac{2}{3}$, the pursuer positions relative to the evader satisfy:

$$y_1' = -\tfrac{2}{3}a_1 + \tfrac{b}{2},\ y_2' = -a_2 + \tfrac{b}{2}\ .$$

The target set $\mathcal{T}$ is the union of the two sets $\{(x_1,x_2)\,|\,|x_i \le r|\}$, $i = 1,2$, for some $r > 0$.

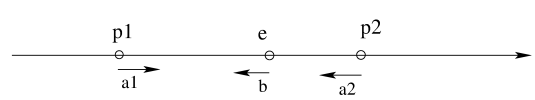

Fig. 1. An unidimensional PE game with two pursuers.

The HJI equation is

$$\begin{cases} u(x) + \max_{a_1,a_2} \min_b \left\{(\frac{2}{3}a_1 - \frac{b}{2}, a_2 - \frac{b}{2})\cdot Du(x)\right\} = 1 \\ \qquad\qquad x \in [0,+\infty]^2 \setminus \mathcal{T} \\ u(x) = 0 \qquad\qquad x \in \partial\mathcal{T} \end{cases}$$

The state dimension is 2. An an analogous problem with m pursuers has state dimension m. The growth in dimensionality places severe restrictions on the computability of solutions to the HJI equation for problems with a large number of pursuers.

III. A DECOMPOSITION TECHNIQUE FOR THE m PURSUER PROBLEM

We now describe a decomposition technique to reduce the complexity of the multiple pursuer game above, for higher state dimensions. The target set $\mathcal{T}$ is decomposed into a union of smaller sets $\{\mathcal{T}_i\}$:

$$\mathcal{T}_i = \{(x_1, x_2, ..., x_m) \in \mathbb{R}^N : |x_i| \leq r\},\ i = 1, \ldots, m.$$

and the value function $u(.)$ is related to the value functions $u_i(.)$ for the games in which the $\mathcal{T}_i$'s replace $\mathcal{T}$. The $u_i(.)$'s are viscosity solutions to the HJI equation above, with modified boundary condition:

$$\begin{cases} u_i(x) + H\left(x, Du_i(x)\right) = 0 & x \in \mathbb{R}^N \setminus \mathcal{T}_i \\ u_i(x) = 0 & x \in \mathcal{T}_i \end{cases} \tag{3}$$

The $u_i(.)$'s are Lipschitz continuous functions.

Define the index set

$$I(x) \ := \ \{i \in \{1, \ldots, m\} : u_i(x) = \min_i u(i)\} \ .$$

Theorem 3.1: Assume condition (C) is satisfied:

(C): for arbitrary $x \in \mathbb{R}^N \backslash \mathcal{T}$, any convex combination $\{\lambda_i \,|\, i \in I(x)\}$ and any collection of vectors $\{\xi_i \in \partial^L u_i(x) \,|\, i \in I(x)\}$ we have

$$H(x, \sum_i \lambda_i \xi_i) \ \leq \ \sum_{i \in I(x)} \lambda_i H(x, \xi_i) \, .$$

Then, for all $x \in \mathbb{R}^N \backslash \mathcal{T}$,

$$u(x) \ = \ \min_i \{u_1(x), \ldots, u_m(x)\} \, .$$

Remark 3.2: Here the limiting superdifferential $\partial^L u_i(x)$ is the set

$$\partial^L u_i(x) \ := \ \limsup_{x' \to x} \partial^F u_i(x) \ ,$$

in which $\partial^F u_i(x)$ is the super Frećhet differential

$$\partial^F u_i(x) \ = \ \{p \in \mathbb{R}^N : \limsup_{x' \to x} \tfrac{u_i(x') - u_i(x) - p \cdot (x' - x)}{|x' - x|} \leq 0\} \, .$$

Condition (C) is automatically is satisfied if $H(x, .)$ is a convex function, but is in fact significantly weaker.

Proof (Outline). Define

$$\overline{u}(x) := \min\{u_i(x) \,|\, i \in 1 \ldots m\} \quad \text{for all } x$$

Since the HJI equation is a unique viscosity solution for the given boundary condions, which are satisfied by $\overline{u}(.)$, it suffices to show that $\overline{u}(.)$ is such a solution.

That $\overline{u}(.)$ is a super (viscosity) solution follows directly from the definition of super solution and the fact that $\overline{u}(.)$ is the lower envelope of a finite number of super solutions. It suffices to demonstrate that

$$\overline{u}(x) + H(x, \xi) \ \leq \ 0 \ . \tag{4}$$

for arbitrary $x \in \mathbb{R}^N \backslash \mathcal{T}$ and any $\xi \in \partial^F \bar{u}(x)$. By the max rule for limiting subdifferentials of Lipschitz functions, applied to $-\overline{u}(x) = \max_i \{-u_i(x)\}$,

$$\xi = \sum_{i \in I(x)} \lambda_i \xi_i \ ,$$

for some convex combination $\{\lambda_i \,|\, i \in I(x)\}$ and vectors $\{\xi_i \in \partial^L u_i(x) \,|\, i \in I(x)\}$. For each $i \in I(x)$, there exist sequences $x_j^i \to x$ and $\xi_j^i \to \xi_i$ such that, for each i, $\xi \in \partial^F u_i(x_j^i)$ as $j \to \infty$. Since $u_i(.)$ is a subsolution, $u_i(x_j^i) + H(x_j^i, \xi_j^i) \leq 0$. By continuity,

$$u_i(x) + H(x, \xi_i) \leq u_i + \limsup_j H(x_j^i, \xi_j^i) \ \leq 0 \, .$$

It follows from condition (C) that

$$H(x, \xi) \ = \ H(x, \sum_i \lambda_i \xi_i) \ \leq \ \sum_i \lambda_i H(x, \xi_i) \ .$$

Then

$$\begin{aligned} \overline{u}(x) + H\left(x, \sum_i \lambda_i p_i\right) &\leq \sum_i \lambda_i \overline{u} + \sum_i \lambda_i H(x, p_i) \\ &\leq \overline{u}(x) + \sum_i \lambda_i \left(u_i + H(x, p_i)\right) \leq 0. \end{aligned}$$

We have confirmed (4). □

We now discuss the role the decomposition technique, summarized as Theorem 3.1, in reducing the complexity of the differential game.

Consider again Example 1. The target can be expressed as a union of two sets: $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$, in which $\mathcal{T}_i := \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq r\}$. For the target $\mathcal{T}_1$ (respectively $\mathcal{T}_2$), the value function is clearly independent of x_2 (respectively x_1). So $\frac{\partial}{\partial x_2} u_1(x_1, x_2) = 0$ and $\frac{\partial}{\partial x_1} u_2(x_1, x_2) = 0$. $u_1(.)$ therefore satisfies

$$\begin{cases} u_1 + \max_{a_1} \min_b \left\{-(-\frac{2}{3} a_1 + \frac{b}{2}) \cdot \frac{\partial}{\partial x_1} u_1\right\} = 1 \\ \qquad\qquad\qquad\qquad x_1 \in (r, +\infty], x_2 \in [0, +\infty) \\ u_1 = 0 \qquad\qquad\quad x_1 \in [0, r], x_2 \in [0, +\infty) \end{cases} \tag{5}$$

where $a_1 \in \mathcal{S}_n(0, \rho_a)$ and $b \in \mathcal{S}_n(0, \rho_b)$. This is a 1D equation for a fixed $\overline{x}_2$ and is constant for a fixed $\overline{x}_1$. It has solution

$$u_1(x) = 1 - e^{-6(x_1 - r)}. \tag{6}$$

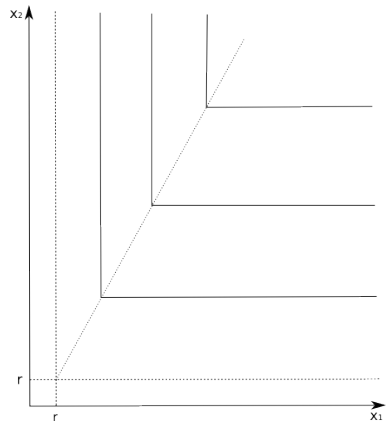

Fig. 2. Some level sets for the minimum time function (Kruzkov's anti transform of the value function) of the unidimensional problem with two pursuers

Similarly

$$u_2(x) = 1 - e^{-2(x_2 - r)}.$$

The gradients of these two functions at x are of the form $(k_1(x), 0)$, $(0, k_2(x))$ for non-negative functions $k_1(.)$ and $k_2(.)$. It is simple to check that condition (C) is satisfied. According to Thm. 3.1, the value function is the lower envelope of $u_1(.)$ and $u_2(.)$, thus

$$t_x = \psi^{-1}(\overline{u}) = \begin{cases} -6(x_1 - r) & \text{if } x_1 \leq \frac{1}{3}x_2 + \frac{2}{3}r \\ -2(x_2 - r) & \text{if } x_1 > \frac{1}{3}x_2 + \frac{2}{3}r. \end{cases}$$

Remark 3.3: When one pursuer is very close to the evader, that pursuer's location alone affects the strategy of the evader, consistent with intuition. This feature of the solution is evident from the formulae for the value function, which reveal that, for states far from the x_1 axis but close to the x_2 (for example), the value function coincides with the function $u_2(x)$.

The preceding analysis can be generalized to cover a unidimensional problem with m pursuers.

Theorem 3.4: Assume hypotheses (H) and additionally, that $g_i(x) \equiv g_i(x_i)$, $h(x) \equiv h(x_i)$, $l_i(x) \equiv l_i(x_i)$ (speeds and costs of the i-player, depend just on the system i-player and evader). Denote by $v_i(.) : \mathbb{R} \to \mathbb{R}$ the solution of the following equation

$$\begin{cases} v_i(x_i) + \max\limits_{a_i} \min\limits_{b} \{f_i(x_i, a_i, b) \cdot Dv_i(x_i)\} = 1 & \\ & x_i \in (r, +\infty] \\ v_i(x_i) = 0 & x_i \in [0, r], \end{cases}$$

where $f_i(x_i, a_i, b) := g_i(x_i)a_i - h(x_i)b - l_i(x_i)$, $a_i \in \mathcal{S}_1(0, \rho_a)$, $b \in \mathcal{S}_1(0, \rho_b)$.

$$u_i(x = (x_1, x_2, ..., x_i, ..., x_m)) = v_i(x_i).$$

Then the value function $u(.)$ is

$$u(x) \;=\; \min\{u_1(x), \ldots, u_m(x)\}.$$

Proof: It is straightforward to confirm that $u_i(x)$ is a viscosity solution of the decomposed problem, using the fact that $\frac{\partial u_i}{\partial x_j}(x) = 0$ for all $i \neq j$. Then, in view of the uniqueness of viscosity solutions ([1], Theorem 3.1), we know that $u_i(.)$ is the unique viscosity solution of the decomposed problem (3).

Making use of the fact that that $\xi_i \in \partial^L u_i(x)$ has the structure $(0, \ldots, 0, k_i, 0, \ldots, 0)$, where $k_i \geq 0$, we deduce then, for any $\xi_i \in \partial^L u_i(x)$ and $\xi_j \in \partial^L u_j(x)$ with $i, j \in I(x)$, we have

$$\begin{aligned} &H(x, \xi_i + \xi_j) = \\ &\rho_a\left(g_i|k_i| + g_j|k_j|\right) - \rho_b|h_ik_i + h_jk_j| - l_i \cdot (0...k_i...k_j...0) \\ &\quad = (\rho_a g_i|k_i| - \rho_b|h_ik_i| - l_i \cdot (0...k_i...0)) \\ &\quad + (\rho_a g_j|k_j| - \rho_b|h_jk_j| - l_j \cdot (0...k_j...0)) \\ &\quad = H(x, \xi_i) + H(x, \xi_j). \end{aligned} \tag{7}$$

This confirms condition (C) of Thm. 3.1 is satisfied. Now apply Thm. 3.1. □

IV. Examples and numerical tests.

In this section we solve some higher dimensional problems using the proposed decomposition technique. Memory storage requirements severely limit the state dimension for direct solution of the associated HJI equation: MATLAB implementations using a heap-based Java VM system are not feasible for N dimensional arrays, for $N > 5$. This motivates the decomposition techniques presented here.

Test 1: Consider the problem when the variable x_i is the distance between the evader and the i'th pursuer and every pursuer has constant maximum speed $g_i(x) \equiv 1$. The evader has constant maximum speed $h(x) = 0.9$. Figs. 3, 4 we show simulations for two different starting points.

Test 2: Now consider several pursuers and one evader in the presence of an obstacle which affects the velocities of the pursuers and the evader. In this case the first m block components of the state are associated with the pursuers and the $(m+1)$'th block component with the evader. Their positions in $\mathbb{R}^n$, satisfy

$$\begin{cases} y_1'(t) = -g(y_1(t))a_1(t) \\ y_2'(t) = -g(y_2(t))a_2(t) \\ \ldots \\ y_m'(t) = -g(y_m(t))a_m(t) \\ y_{m+1}'(t) = h(y_{m+1}(t))b(t)\,, \end{cases}$$

where the state $y := (y_1^T, y_2^T, ..., y_{m+1}^T)^T \in \mathbb{R}^N$ and every $y_i \in \mathbb{R}^n$. We assume that $g : \mathbb{R}^n \to \mathbb{R}^+$ and $h : \mathbb{R}^n \to \mathbb{R}^+$, so every pursuer has the same velocity rule

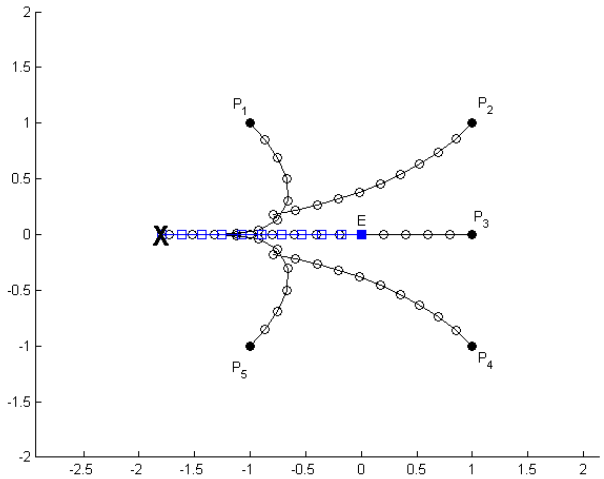

Fig. 3. Test 1: optimal trajectories at various times. A X indicates the point of capture.

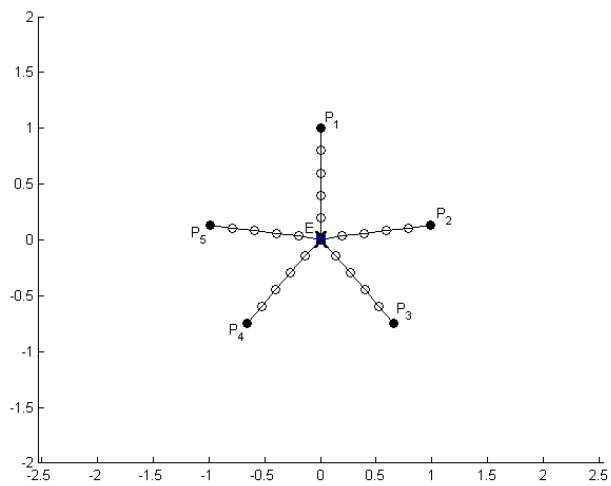

Fig. 4. Test 1: optimal trajectories at various times. A X indicates the point of capture. The optimal trajectory of the evader is waiting the capture.

(depending on the position) This feature will permit decomposition. The target set is

$$\mathcal{T} = \Big\{(x_1^T, x_2^T, ..., x_{m+1}^T) \in \mathbb{R}^N : \min_{i\in\{1,2...m\}} |x_i - x_{m+1}| \leq r \Big\}$$

Fig. 5 illustrates the target set for $m = 2$, $n = 1$.

The Hamiltonian, which is

$$H(x,p) = \sum_{i=1}^{m} g(x_i)\rho_a|p_i| - h(x_{m+1})\rho_b|p_{m+1}| - 1\,,$$

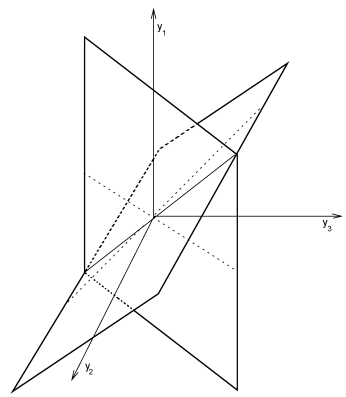

Fig. 5. Test 2: Target set for $m = 2$, $n = 1$, $r = 0$.

is not convex in the p variable.

Theorem 4.1: Consider the PE game with dynamics and constraints defined by (1) and (2). Let $u_i(.) : \mathbb{R}^n \to \mathbb{R}$ the solution of the following equation

$$\begin{cases} v_i(x_i) + \max_{a_i} \min_{b} \{(g(x_i)a_i, -h(x_{m+1})b) \cdot Dv_i(x_i)\} = 1 & x_i \in \{x_i : |x_i - x_{m+1}| > r\} \\ v_i(x_i) = 0 & x_i \in \{|x_i - x_{m+1}| \leq r\} \end{cases}$$

with $a_i \in \mathcal{S}_n(0, \rho_a)$, $b \in \mathcal{S}_n(0, \rho_b)$. Define the function $u_i(x) : \mathbb{R}^N \to \mathbb{R}$ as

$$u_i(x = (x_1, \ldots, , x_m)) = v_i(x_i)\ .$$

Then the value function for the original problem is

$$u(x) = \min\{u_1(x), \ldots, u_m(x)\}.$$

Proof: (Outline). Suppose that $n = 1$. (General n is treated similarly). The Hamiltonian is convex in p along rays in every direction except the e_{m+1} direction. (Here, e_i is i'th canonical basis vector).

We can repeat the main steps in the proof of Theorem 3.4 with the exception of the verification of condition (C). In this case elements in the limiting superdifferential are aligned with e_{m+1}, thus

$$\xi_i, \xi_j \in \partial^F u(x), \quad (\xi_i - \xi_j) \cdot e_{m+1} = 0.$$

This follows from the fact that e_{m+1} is tangential to the switching interface. Writing u_i and u_j for two reduced value functions, we can show that $u_i(.) = u_j(.)$, i.e. they solve the same equation. It follows that the switching interface is located where two reduced value functions coincide, i.e. where $u_j(x) = u_i(x)$. For n the normal of the switching interface,

$$n \cdot e_{m+1} = \left(0, \ldots, 0, \frac{\partial u_i}{\partial x_i}, 0, \ldots, 0, \frac{\partial u_i}{\partial x_j}, 0, \ldots, 0\right) \cdot (0, \ldots, 0, 1)^T = 0 \,. \quad (8)$$

Condition (C) can now be validated. the state representation of $u(.)$ is therefore valid by Thm. 3.1. □

As an example, consider the $n = 2$ case. An evader has position denoted by $x_e \in \mathbb{R}^2$ and $m \in \mathbb{N}$ pursuers have positions denoted by $x_1, x_2, ...x_n \in \mathbb{R}^2$. Take $m = 3$. It is assumed that for a $x_i = (x_{i,1}, x_{i,2})$

$$\begin{cases} g(x_i) = 1 - 0.5\cos(\pi x_{i,2}) & \text{if } |x_{i,2}| < 0.5, \\ g(x_i) = 1 & \text{elsewhere,} \end{cases} \quad (9)$$

for all $i = 1, 2, 3$; and $h(x_e) \equiv 0.4$.

Figs. 6, 7 show some optimal trajectories and the level sets of the velocity function of the pursuers.

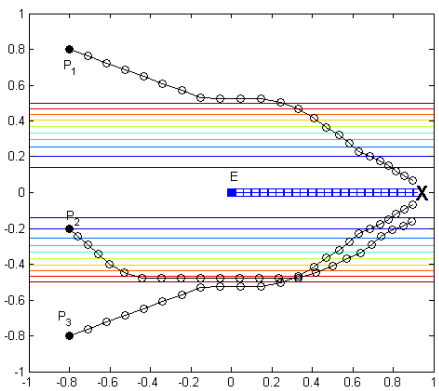

Fig. 6. Test 2: optimal trajectories for a 3pursuers game and level sets of the function g (maximum speed of the pursuers).

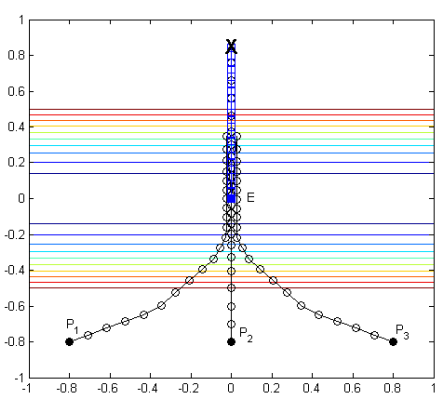

Fig. 7. Test 2: optimal trajectories for a 3pursuers game and level sets of the function g (maximum speed of the pursuers).

V. Conclusion

An abstract decomposition technique has been presented, for reducing the complexity of PI games with many pursuers in certain cases. Related research themes currently under investigation include the verification of the condition (C), justifying the decomposition, in more general situations and the extension of the preceding analysis with many pursuers and evaders.

VI. Acknowledgments

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References

[1] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhäuser, Boston 1997.

[2] M. Bardi, T.E.S. Raghavan, T. Parthasarathy, *Stochastic and Differential Games: Theory and Numerical Methods*, Birkhäuser, Boston, 1999.

[3] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Birkhäuser, Boston, 2004.

[4] W. Chodun, Differential games of evasion with many pursuers, *J. Math. Anal. Appl.* 142 (1989) no. 2, 370–389.

[5] R.J. Elliott and N.J. Kalton. Values in differential games. *Bull. Amer. Math. Soc.* Vol. 78 (1972), no. 3, 427–431.

[6] G.M. Erickson, A differential game model of the marketing-operations interface, *Eur. J. Oper. Res.* 211 (2011) 294–402.

[7] P. Falugi, C. Kountouriotis and R. Vinter, Differential Games Controllers That Confine a System to a Safe Region in the State Space, With Applications to Surge Tank Control, *IEEE T. Automat. Contr.* 57 (2012) no. 11, 2778-2788 .

[8] A. Friedman, *Differential Games*, John Wiley & Sons, New York, USA, 1971.

[9] J. Hu and M. Wellman. Multiagent Reinforcement Learning: Theoretical Framework and an Algorithm. *Proceedings of ICML*, (1998) 242-250.

[10] G.I. Ibragimov, Optimal pursuit of an evader by countably many pursuers. *Differ. Equ.* 41 (2005) no. 5, 627-635.

[11] R. Isaacs, *Differential Games*, John Wiley & Sons, New York, USA, 1965.

[12] R.P. Ivanov and Yu. S. Ledyaev, Time optimality for the pursuit of several objects with simple motion in a differential game. *Trudy Mat. Inst. Steklov.* 158 (1981), 87–97.

[13] S. Jørgensen, Optimal production, purchasing and pricing: A differential game approach, *Eur. J. Oper. Res.* 24 (1986) no.1, 64–76.

[14] N.N. Krasovskii and A.I. Subbotin, *Game-Theoretical Control Problems*, Springer, New York, 1988.

[15] L.S. Pontryagrin, *Izbrange Trudy (Selected works)*, Moskow, Russia, 1988.

[16] B.N. Pshenichnii, Simple pursuit by several objects, *Cybern. Syst. Anal.* 12 (1976) no.3, 484–485.

[17] D. M. Stipanovic, A. Melikyan and N. Hovakimyan, Guaranteed Strategies for Nonlinear Multi-Player Pursuit-Evasion Games, *Int. Game Theor. Rev.*, 12 (2010) no. 1, 1-17.

[18] R. Vidal, O. Shakernia, J. Kim, Associate Member, H. Shim, S. Sastry, Probabilistic Pursuit–Evasion Games: Theory, Implementation, and Experimental Evaluation, *IEEE T. Robotic. Autom.*, 18 (2002) no. 5, 662–669.