EE2 Maths Summary Sheet: Functions of Multiple Variables

1) The change in f following a small step $\Delta x, \Delta y$ is:

$$\Delta f \approx \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y. \tag{1}$$

The error in this approximation goes to zero as $\Delta x, \Delta y \to 0$.

2) In the limit $\Delta x, \Delta y \to 0$ Eq. (1) becomes the **Total Differential**:

$$\boxed{df = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy} \tag{2}$$

This generalises to $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots \frac{\partial f}{\partial x_n}dx_n$ for functions of n variables.

3) Sometimes we have $f(x,y)$ but x and y are both themselves functions of a variable u such that $x(u), y(u)$. In this case the **Chain Rule** applies:

$$\boxed{\frac{df}{du} = \frac{\partial f(x,y)}{\partial x}\frac{dx}{du} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{du}.} \tag{3}$$

Note that this only applies when f is a univariate function of u.

4) Sometimes we can express $f(x,y)$ in another co-ordinate system as $f(u,v)$. In this case we can relate the co-ordinate systems by $x = x(u,v)$ and $y = y(u,v)$. We can then relate the partial differentials of f as follows:

$$\left.\frac{\partial f}{\partial u}\right|_v = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial u}\right|_v + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial u}\right|_v \tag{4}$$

$$\left.\frac{\partial f}{\partial v}\right|_u = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial v}\right|_u + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial v}\right|_u \tag{5}$$

A rule of thumb to obtain e.g. Eq. (5) is to take the total differential Eq. (2) and "multiply through by $\left.\frac{}{\partial v}\right|_u$". Equations like Eq. (4,5) are sometimes referred to as *"expressing a change of variables in f"*.

5) A Taylor's series expansion about a point can be generalized to the multivariate case. Below is the expression for an expansion of $f(x,y)$ about (x_0, y_0) where we define $\Delta x = x - x_0$ and $\Delta y = y - y_0$ and evaluate the derivatives at x_0, y_0:

$$f(x,y) = f(x_0,y_0) + \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right] + \ldots \tag{6}$$

6) The total differential Eq. (2) can be written as $df = d\underline{s} \cdot \nabla f$ where $d\underline{s} = (dx, dy)$ and $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is called the *Gradient Vector*. ∇f: points, in the $x-y$-plane, in the direction of greatest change of f. It's magnitude $|\nabla f(x,y)|$ tells us the gradient in the direction $\nabla f(x,y)$.

7) There are three types of stationary points of $f(x,y)$: Maxima, Minima and Saddle Points. The sufficient conditions for a stationary point to be a Max, Min, Saddle are:

$$Max: \quad f_{xx} < 0 \quad \text{and} \quad f_{xy}^2 - f_{xx}f_{yy} < 0 \tag{7}$$

$$Min: \quad f_{xx} > 0 \quad \text{and} \quad f_{xy}^2 - f_{xx}f_{yy} < 0 \tag{8}$$

$$Saddle: \quad f_{xy}^2 - f_{xx}f_{yy} > 0. \tag{9}$$

8) If $f(x_1...x_n)$ we can define $H_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f(x_1,...,x_n)$. The matrix H is called the *Hessian*.

9) Given that

$$F(x) = \int_{t=u(x)}^{t=v(x)} f(x,t)dt \tag{10}$$

the Leibnitz' Integral Rule is then:

$$\frac{dF(x)}{dx} = f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx} + \int_{t=u(x)}^{t=v(x)} \frac{\partial f}{\partial x}dt. \tag{11}$$