Collision avoidance in pedestrian dynamics

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***Abstract*— Karamouzas et al. [21] presented strong evidence that collision avoidance is one of the main interaction rules in pedestrian dynamics. In other words individuals actively anticipate the future to predict a possible collision time and adjust their velocity accordingly. However they show no intention to change their direction when walking close to each other in the same direction. This collision avoidance behavior initiates complex dynamical phenomena, an assumption confirmed by analyzing experimental data and reproduced in numerical simulations. The simulations show that already a simple dynamical system model, with forces depending on the estimated hitting time, reproduces complex dynamics such as velocity alignment.**

In this paper we propose an optimal control model, that is based on the idea of a two-player pursuer evader game. We shall use Bellman's approach to study the embedded game for collision avoidance and discuss related theoretical as well as numerical aspects.

I. Introduction

Pedestrian dynamics can be described on various scales: on the microscopic level by specifying the motion of every single individual according to certain interaction rules between them and their environment. On the meso- and macroscopic level by considering the distribution of all particles with respect to their position and velocity and describing their evolution using differential equations. The latter approach is mathematically more amenable, but only valid for large pedestrian groups. Microscopic models allow for the description of fewer individuals, but analyzing the overall dynamics is often not possible. The dynamics are driven by specifically stated interaction laws and are therefore more intuitive. Depending on the situation considered one has to weigh the pros and cons of either level to select the appropriate description.

On the microscopic level force based models, such as the social force model by Helbing [17], are a popular choice. These models correspond to first or second order dynamical systems, in which the position and velocity change according to given interaction forces. In differential games these forces correspond to the solution of an optimal control problem - each individual is assumed to be rational, basing its optimal strategy on minimizing a given cost functional. These cost may relate to the kinetic energy, exit time or collision avoidance, cf. e.g. [18], [11].

In mesoscopic models the evolution of the distribution of individuals with respect to their position and velocity is given. Interactions between agents are modeled by 'collisions', see for example [25]. Macroscopic models are often based on conservation laws, describing the evolution or change of certain quantities such as the density or momentum in time, cf. [26], [13]. Recently mean field game approaches, cf. [19], [23], have been proposed to model the evolution of large pedestrian crowds, see [22]. These models are closely related to parabolic optimal control approaches, in which the optimal strategy of the crowd is determined by minimizing a given cost. For a detailed overview on different modeling approaches in pedestrian dynamics we refer to [8].

Experiments indicate that repulsion between individuals is not only related to their physical distance - two pedestrians walking with similar speed in the same direction effect one another less than two individuals moving towards each other. This intuitive assumption was recently confirmed by Karamouzas et al in [21]. In addition to their experimental findings they present a force based model, in which repulsion between individuals is directly related to the estimated collision time. The estimated collision time is calculated by extrapolating the actual trajectories and speeds in time.

In the present work we propose a different microscopic approach to incorporate collision avoidance behavior in pedestrian dynamics. We assume that individuals act rationally by minimizing/maximizing certain costs. The costs are related to the anticipated collision time. This problem corresponds to an optimal control problem, in which the dynamics of each individual are given by a second order dynamical system. We consider the case of *non anticipating strategies*; i.e. each individual knows its current position and velocity only and adjusts them with respect to the current states of all other individuals. These methods are well known in *differential games*, see [20], [7]. They provide a global technique for finding the value function of the problem (which is the potential linked to the optimal strategies of the players) using the corresponding Hamilton-Jacobi-Isaacs equation. Recently these techniques were proposed to solve collision avoidance problems in

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traffic flow problems, see for example [24], [28]. The advantage of this approach is the global optimality and the possibility to compute *off line* solution due to the open-loop nature of the control. On the other hand a well known challenge in these problems is the 'curse of dimensionality'. For example in the multi-player case, the optimal strategy for each player is obtained by solving a partial differential equation in a domain of dimension $4d \times (n-1)$, where d is the space dimension and n the number of players. In this paper we propose a sub-optimal technique, namely each individual adjusts its position and velocity by calculating the optimal strategy with respect to the estimated collision time with respect to every other individual separately.

This paper is organized as follows: in section II we present a collision avoidance framework and its mathematical background. Section III discusses the integration of boundaries and additional objectives in the model. Numerical experiments illustrate the complex behavior of the proposed model in section IV. We conclude by discussing our findings in section V.

II. Collision avoidance

We start by discussing collision avoidance strategies for differential game models. We consider the situation, in which each player bases its decision on the actual state of all other players. This corresponds to a *non-anticipating strategy*, see [14] as well as [6], [5] for more details. We shall study the behavior of the system in case of general and reduced dynamics of a single agent and discuss the notion of optimality in this context.

A. General dynamics

Let us consider an $n-$agent system described by the following state equations:

$$\begin{cases} \dot{x}_i = f(x_i, v_i) \\ \dot{v}_i = g(x_i, v_i, a_i), \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) = z, \end{cases} \tag{1}$$

where $i \in \{1, ..., n\}$ denotes the $i-$th player with position and velocity $(x_i, v_i) \in \mathbb{R}^{2d}$. Throughout this paper we make the following assumptions for the functions f and g:

$$(H1) \quad \left.\begin{array}{l} f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \\ g : \mathbb{R}^d \times \mathbb{R}^d \times A \to \mathbb{R}^d \end{array}\right\} \text{ are continuous}$$
$$\text{and Lipschitz continuous w.r.t. } (x_i, v_i).$$

The parameter $a_i \in A$ is the *control* chosen from the compact set A. In the following we refer to $y_i(t) := (x_i(t), v_i(t))$ as the solution to system (1) for a control $a_i = a_i(t)$.

We start by discussing the optimal collision avoidance strategy in the case of two players, i.e. $n = 2$ and we shall use the notation P1 and P2 to refer to either one of them. In this simple situation P1 determines its optimal strategy given the dynamics of P2. Its objective corresponds to staying within a certain distance to player P2. Hence we define the set $\mathcal{G}$:

$$\mathcal{G} := \left\{ y = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d \text{ s.t. } \|x\|_2 \le \rho \right\},$$

where $\rho > 0$ denotes to the minimal distance P1 wants to maintain. Then the corresponding optimal control problem can be written as follows: given two sets of initial data $y_1(0)$ and $y_2(0)$ determine

$$\sup_{\varphi \in \Phi} \inf_{a_2 \in A} \int_0^{+\infty} e^{-\lambda t} \chi_{\mathcal{G}}(y_1(t) - y_2(t), t)\, dt\,, \tag{2}$$

subject to (1).

In (2) the parameter $\lambda > 0$ denotes the temporal *discount factor* and the function χ is defined by

$$\chi_{\mathcal{G}}(y(t), t) := \begin{cases} 0 & \text{if } \exists \bar{t} \le t \text{ s.t. } y(\bar{t}) \notin \mathcal{G} \\ 1 & \text{elsewhere.} \end{cases}$$

Note that $\min\{t | \chi_{\mathcal{G}}(y(t), t) = 0\}$ corresponds to the first time when the trajectory $y(t)$ enters the set $\mathcal{G}$. The space Φ is the closed-loop functional space of the *non-anticipating strategies* (chosen by P1), in response to the optimal strategy by P2. The space of the control functions is set to:

$$\mathcal{A} := \{ a(\cdot) : [0, \infty) \to A \,|\, a(.) \text{ meas. and } a(t) \in A \text{ a.e. } \}.$$

The strategy space Φ is defined as

$$\Phi := \{ \varphi : \mathcal{A} \to \mathcal{A} \,|\, \varphi \text{ is non-anticipative} \}.$$

We shall use the following definition for non-anticipative behaviour: for any $t' \ge 0$, and two given controls $\bar{a}_2(.)$ and $\hat{a}_2(.) \in \mathcal{A}$,

$$\bar{a}_2(t) = \hat{a}_2(t) \text{ a.e. } t \in [0, t'] \implies \varphi(\bar{a}_2(.))(t) = \varphi(\hat{a}_2(.))(t) \text{ for a.e. } t \in [0, t'].$$

Then the corresponding value function of (2) (obtained using classical results from optimal control theory, cf. [3]) is given by

$$u(z) = \sup_{\varphi \in \Phi} \inf_{a_2 \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda t} \chi_{\mathcal{G},t}(y_1(t) - y_2(t), t)\, dt\,,$$

is the viscosity solution of the following Hamilton-Jacobi-Isaacs equation:

$$\begin{cases} \lambda u(z) + \min\limits_{a_1 \in A} \max\limits_{a_2 \in A} \{ -(f(x_1, v_1), g(x_1, v_1, a_1) \\ \qquad f(x_2, v_2), g(x_2, v_2, a_2)) \cdot Du(z) \} = 1 \\ \qquad\qquad z \in \mathbb{R}^{4d} \setminus \mathcal{G}, \\ u(z) = 0, \qquad z \in \mathcal{G}. \end{cases} \tag{3}$$

The existence of a viscosity solution for this equation is guaranteed by hypothesis H1. To ensure uniqueness we need the additional assumption that

(H2) The viscosity solution u of (3) is continuous in $\partial\mathcal{G}$.

Then player P1 determines its the best strategy by assuming that it knows the actual position and velocity of P2 (but not the future). Such control can be expressed as a feedback map (open-loop control) depending by the state of the system composed by the two players. Hence its control is given by

$$a_1(z) \in \arg\min \max_{a_2}\{-(f(x_1, v_1), g(x_1, v_1, \cdot), f(x_2, v_2), g(x_2, v_2, a_2)) \cdot Du(z)\}. \quad (4)$$

The design of the control follows the so called *synthesis problem* related to differential games. The selected choice is optimal to avoid collisions and satisfies the Pontryagin's optimality conditions , cf. [10], [27]. Note that the value function of a differential game corresponds to a *Nash equilibrium* [7]: the control is the best answer to the choice of its opponent. In (1) we considered the case of indistinguishable players, which follow the same dynamics f and g. Note that this analysis can be easily generalized (for the cost of readability) to describe agents with different dynamical features, such as pedestrians with different speeds, cyclists or cars.

In the case of multiple agents we use the solutions of each separate $2-$player game using the minimal value function, i.e.

$$u_i(z) = \min_j u_{i,j}(z), \quad (5)$$

where $u_{i,j}(z)$ is the solution of the i vs. j player game to (3), $j = 1, \dots n,\ j \neq i$. Hence each player adjusts its velocity with respect to the agent with the smallest collision time. Note that this choice corresponds to a special projection of the high-dimensional solution (of the multi-agent problem), which *may not be optimal*. The discussion about optimality in the case of multi-player dynamics is out of the scope for this short paper, we refer to [16] for further details.

B. Reduced dynamics

Next we consider system (1) in a special case of reduced dynamics. This problem is mathematically more amendable and shall serve as an educational example to understand the underlying dynamics and relate optimal control approaches to force based models. Hence we study:

$$\dot{y}_i = \begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = a_i. \end{cases} \qquad A := B_d(0, 1). \quad (6)$$

Here $B_d(0, 1)$ denotes the closed ball centered around the origin with radius 1. In this case we can calculate the value function of (3) analytically.

Again we consider the interactions between two players only ($i-$player and $j-$player). Due to the special structure of the dynamics in (6), i.e. the dynamics do not depend on the position of each player, we can study the system with respect to the relative position of the agents (*reduced dynamics* in [4]). Let

$$s := x_i - x_j \text{ and } q := v_i - v_j.$$

Then (3) reduces to

$$\lambda u(s, q) + \min_{a_i \in A} \max_{a_j \in A}\{-(q, a_i - a_j) \cdot Du(s, q)\} = 1.$$

In this case the computation of the nonlinear part corresponds to

$$\lambda u(s, q) - (q, 0) \cdot Du(s, q) = 1$$

where $Du := (\nabla_s u, \nabla_q u)$ and $\nabla_s u$ denotes the components of the gradient with respect to s. Hence we have

$$-q \cdot \nabla_s u = 1 - \lambda u.$$

With the notation $|\cdot| := \|\cdot\|_2$, a Lipschitz continuous solution of such equation is given by

$$u(s, q) := \begin{cases} \frac{1}{\lambda}(1 - e^{-\lambda\tau(s,q)}) \\ \qquad \text{if } \exists t > 0 \text{ s.t. } |s + tq| \leq \rho \\ 0 \qquad\qquad \text{if } |s| \leq \rho \\ \frac{1}{\lambda} \qquad\qquad \text{elsewhere,} \end{cases}$$

where $\tau(s, q) := -\frac{s\cdot q + \sqrt{(s\cdot q)^2 - |q|^2(|x|^2 - \rho^2)}}{|q|^2}$. Note that an explicit collision condition equivalent to the one stated above is

$$\rho \geq \sqrt{|x|^2 - \frac{(x \cdot v)^2}{|v|^2}}. \quad (7)$$

The solution u is also the unique viscosity solution, if the theoretical assumptions (H1-H2) are satisfied. In this case the control corresponds (independently of the opponent player) to

$$a_i := -\frac{\nabla_q u}{|\nabla_q u|},$$

if $|\nabla_q u| \neq 0$. In the case of no collision, i.e. condition (7) is not satisfied, we have $a_i = 0$.

Remark 1: The problem with reduced dynamics (6) is related to the force based model in the original work of Karamouzas et al. [21]. Moreover our framework provides an optimal control interpretation to their approach. Karamouzas et al. suggested that the forces between individuals depend on their estimated collision time. The estimated collision time $t_{i,j}$ of player i and player j is based on the extrapolation of their current trajectories and velocities and the assumption that every player will continue its trajectory along this line. The hitting time can be computed explicitly by the following formula:

$$t_{i,j} = \begin{cases} 0 & \text{if } |x_i - x_j| < \rho, \\ +\infty & \text{if no collision,} \\ -\frac{(x_i-x_j)\cdot(v_i-v_j)+\sqrt{\Delta}}{|v_i-v_j|^2}, & \text{elsewhere.} \end{cases}$$
$$\Delta := ((x_i - x_j)\cdot(v_i - v_j))^2 \\ \qquad -|v_i - v_j|^2(|x_i - x_j|^2 - \rho^2).$$

Then the repulsive forces between the individuals correspond to the gradient of the function

$$E_i := \sum_{j \neq i} \left(-\widehat{k}\, \frac{e^{-\frac{t_{i,j}}{\tau_0}}}{t_{i,j}^p} \right), \tag{8}$$

where $\widehat{k}, p, \tau_0 \in \mathbb{R}^+$ are modeling parameters. Note that there is an evident link between the two models: in the case $n = 2$ the value function u and the potential of the force model E_i (8) are equivalent up to a translation constant if $\lambda = 1/\tau_0$, $\widehat{k} = \tau_0$, $p = 0$. In the case of multiple players, i.e. $n > 2$, the difference in the models corresponds to the choice of the potential. In the optimal control approach it related to the minimum collision time of all players, in the force based model to their sum.

III. Confinement and targets

Spatial and velocity restrictions are natural constraints. Especially in pedestrian dynamics confined spaces such as rooms with exits or velocity bounds are a common assumption. In this section we discuss how to include spatial constraints or how to determine the direction to an exit, which is important in the numerical experiments.
In the following we consider problem (2) on the open domain $\Omega \subset \mathbb{R}^d \times \mathbb{R}^d$. The analytic properties of this problem were studied by various authors, for a detailed discussion we refer to [12], [9].

An important aspect of optimal control problems on bounded domains is the *viability* of the dynamics. A sufficient and necessary condition for the viability on Ω (with a sufficiently smooth boundary $\partial\Omega$) is given by:

$$\sup_{a_i \in A} \inf_{a_j \in A} \{(f(x_i, v_i), g(x_i, v_i, a_i), f(x_j, v_j), \\ g(x_j, v_j, a_j)) \cdot \eta(x_i, v_i, x_j, v_j)\} \leq 0,$$

for every $(x_i, v_i, x_j, v_j) \in \partial\Omega \times \partial\Omega$, where $\eta(x_i, v_i, x_j, v_j)$ denotes the exterior normal to $\partial\Omega \times \partial\Omega$, cf. e.g. [2]. This condition, with (H1) and (H2) is sufficient to guarantee the well posedness of the problem.

Next we discuss how to determine the optimal strategy to reach a target $\mathcal{T}$, for example an exit $\mathcal{T} \subset \partial\Omega$ or a certain position in space $\mathcal{T} \subset \bar{\Omega}$. Then the shortest path to a target, is given by the gradient of the potential $w : \Omega \to \mathbb{R}$ ($\Omega := \mathbb{R}^d \times \mathbb{R}^d$ in the unconstrained case), viscosity solution of the Eikonal equation

$$\begin{cases} \max_{a \in A}\{-(f(x,v), g(x,v,a)) \cdot Dw(x,v)\} = 1 \\ \qquad\qquad\qquad\qquad (x,v) \in \Omega \setminus \mathcal{T} \\ w(x,v) = 0 \qquad\qquad (x,v) \in \mathcal{T}. \end{cases}$$

We reiterate that the Eikonal equation gives the value function of the minimum time problem to reach the *target* $\mathcal{T}$ i.e. the optimal control problem

$$\text{Find } w(y) := \inf_{a \in A} \int_0^{+\infty} \chi_{\{y(t) \notin \mathcal{T}\}}\, dt\,,$$
subject to (1) with initial values $y(0)$.

We propose that each agent determines its strategy as the shortest path to the exit while trying to avoid collisions with the other players, i.e.

$$P_i := w_i(x, v) + k(u_i(x, v)), \tag{9}$$

where k is a model parameter which weighs the relevance of collision avoidance against the optimal exit strategy. Note that each agent can have a different potential w_i, $i = 1, \ldots n$, which may correspond to different objectives.

IV. Numerical experiments

In this section we shall illustrate the behavior of the proposed optimal control approach with various numerical experiments. We shall observe complex dynamics, such as milling or 'freezing' in case of bottlenecks. In classical force based models milling emerges from the balance between short range repulsion and long range attraction. This is not the case in our approach - milling behavior results from collision avoidance and spatial restrictions only.
The numerical simulation are based on the reduced dynamics (6), which are easier from the computational point of view but complex enough to obtain non trivial solutions. The solution of the Hamilton-Jacobi equations is based on the policy iteration algorithm [1] using Semi-Lagrangian approximation schemes [15].

In our first test we consider a group of $n = 200$ pedestrians subject to the confinement potential

$$w(x) = \max(0, x_1^2 + x_2^2 - 1).$$

The initial state of every agent (both position and speed) is chosen randomly in the set $[-3, 3]^2 \times [-1, 1]^2$. We observe mill formation for different sets of parameters, see Figure 1. Mill formation is closely related to the number of the agents, the physical space available and to confining potential. In Figure 2 we illustrate the tendency towards synchronization: let

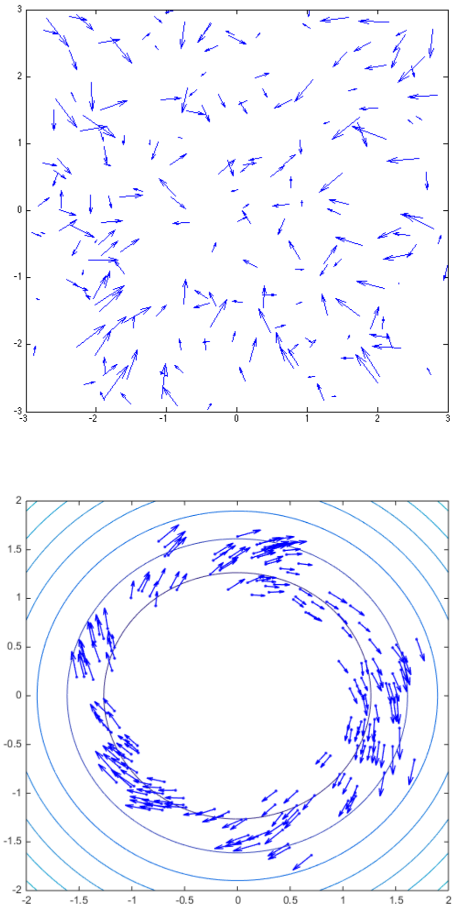

Fig. 1. Milling: initial configuration and state after $t = 30s$ for $k = 2$, $\rho = 0.03$, $\lambda = 1$, $n = 200$.The level sets correspond to the confinement potential.

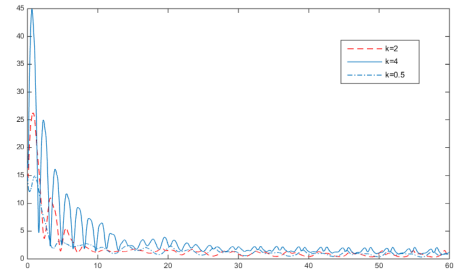

Fig. 2. Evolution of the deviation from the average angular velocity for $n = 200$ individuals.

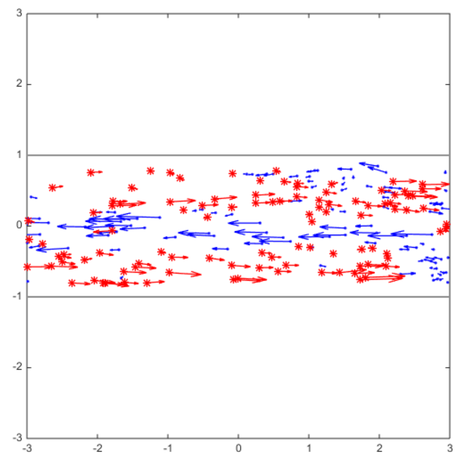

Fig. 3. Formation of directional lanes: state of the system after $t = 10s$, $k = 2$, $\rho = 0.03$, $\lambda = 1$, $n = 200$.

$\dot{\theta}_i(t)$ denote the angular velocity of the $i-$player and $\dot{\theta}_m(t)$ the average over all agents; we evaluate

$$\Sigma(t) := \sum_{i=1}^{n} |\dot{\theta}_i(t) - \dot{\theta}_m(t)|,$$

to study the deviation of the individual angular velocity from the average. In Figure 2 we see that the initially randomized motion aligns along circles and stabilizes with a certain periodicity around this equilibrium state. We compare this behavior for different choices of k; particles align in all three cases, but the time towards stabilization is different.

Another situation of interest is when two population with different objectives are interacting in a restricted space. We model this with two potentials

$$w_1(x) = x_1 + \max(|x_2| - 0.8, 0),$$
$$w_2(x) = -x_1 + \max(|x_2| - 0.8, 0),$$

related to two distinct populations. Note that in our approach collision-avoidance between two agents is completely independent of the belonging to one group or the other. The initial conditions correspond to a random configuration. We assumed periodic conditions on the left and right boundaries of the domain. In this case - Figure 3 - we observe *lane formation*, i.e the formation of directional lanes. The number of lanes depends on the parameter k, i.e. the stronger the collision-avoidance tendency the larger the number of lanes. If we run the simulation with an obstacle in the middle of the domain (simply adding $\max(1 - x_1^2 - x_2^2, 0)$ to potentials w_1 and w_2) the dynamics of the groups converge to two different equilibria, which depend on the initial configuration. These semi-stable configurations are:

1) either group bypasses the obstacle on one side;
2) one population passes the obstacle on both sides, while the other one is confined behind the obstacle. A condition similar to so-called *freezing*.

Situation 2 is shown in Figure 4, where the the red particles bypass the obstacle on both sides while the blue particles hardly move from the right side of the domain.

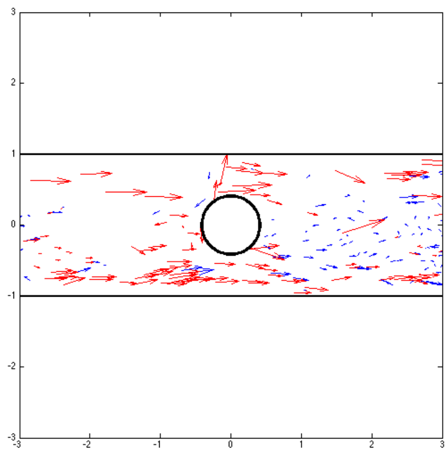

Fig. 4. Congestion configuration caused by an obstacle: state of the system after $t = 10s$, $k = 2$, $\rho = 0.03$, $\lambda = 1$, $n = 200$.

V. Conclusion

In this paper we propose a differential game approach to model collision avoidance in pedestrian crowds. We discuss the general modeling setup and illustrate the behavior of the model with various numerical simulations. These investigations shall serve as a starting point for further research in different directions.

The first is related to the notion of optimality in multi-player games. As discussed in [16] we cannot guarantee optimality in general, but only in situations in which the optimal trajectory does not touch any switching point of the (5). In other words a configuration where the time of hitting with respect of two different agents is coincident and minimal. Another point is to consider general functions f and g to describe more complex situations such as mixed populations, multi-conflicting goals etc. The computational complexity of solving (3) in the case of higher space dimension poses an additional challenge for further research.

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